Math 444 A
Winter 2016

Two versions of the Protractor Postulate

The purpose of this note is to clarify the relationship between the two versions of Axiom 8 in Neutral Geometry (NG from now on) that you've encountered: the one stated in class, and the one in our textbook, on page 85.

I'll first state the version from the textbook, in a form that makes it look as similar to the one from class as possible (so that the distinction becomes apparent). I will use the notation from the book and from class, with $HR(\overrightarrow{r}, P)$ denoting a half-rotation, etc.

Version 1 (**Protractor Postulate**). *For every ray $\overrightarrow{r}$ and every point P not on the line $\overleftrightarrow{r}$ there is a bijection $f : HR(\overrightarrow{r}, P) \to [0, 180]$ such that the following conditions hold:*

- $f(\overleftrightarrow{r}) = 0$*;*
- $f($*ray opposite to* $\overrightarrow{r}) = 180$*;*
- *For every two rays* $\overrightarrow{a}, \overrightarrow{b} \in HR(\overrightarrow{r}, P)$ *we have* $m\angle ab = |f(\overrightarrow{a}) - f(\overrightarrow{b})|$.

On the other hand, in class on Friday (Jan 29) I said this:

Version 2 (**Alternate Protractor Postulate**). *For every ray $\overrightarrow{r}$ and every point P not on the line $\overleftrightarrow{r}$ there is a bijection $f : HR(\overrightarrow{r}, P) \to [0, 180]$ such that the following conditions hold:*

- $f(\overleftrightarrow{r}) = 0$*;*
- *For every two rays* $\overrightarrow{a}, \overrightarrow{b} \in HR(\overrightarrow{r}, P)$ *we have* $m\angle ab = |f(\overrightarrow{a}) - f(\overrightarrow{b})|$.

This makes it clear what the difference was: we were missing the second bullet point. The point I was trying to make in class when I mentioned this though (Monday, Feb 01) was that the distinction is an illusion: we're not losing any of the power of the axiomatic system when dropping that second condition.

More precisely, think of it this way. I seemingly have two axiomatic systems so far: one consisting of axioms (1) - (7) plus Version 1 (call it Partial Neutral Geometry 1 or PNG1 for short), and one consisting of axioms (1) - (7) together with Version 2 instead (called PNG2). A priori, PNG1 is stronger: Version 1 of the 8th axiom clearly stipulates more than Version 2, so every theorem of PNG2 will still be a theorem of PNG1. What I was saying, however, was that the two axiomatic systems are actually equally strong:

Theorem 1. *In PNG2, every bijection $f : HR(\overrightarrow{r}, P) \to [1, 180]$ as in Version 2 of the axiom automatically assigns the value* 180 *to the ray opposite to* $\overrightarrow{r}$.

Proof. Denote by $\overrightarrow{s}$ the ray opposite to $\overrightarrow{r}$, and by H the half-rotation $HR(\overrightarrow{r}, P)$. We will prove the claim by contradiction: suppose I can find a bijection $f : HR(\overrightarrow{r}, P) \to [0, 180]$ satisfying the conditions from Version 2 but such that $f(\overrightarrow{s})$ is *not* 180.

In that case, $f(\overrightarrow{s})$ will be a number in $[0, 180)$. Moreover, since $f(\overrightarrow{r}) = 0$ and $\overrightarrow{s} \neq \overrightarrow{r}$, the injectivity of f implies that $f(\overrightarrow{s})$ belongs to the open interval $(0, 180)$.

Now, since every ray $\overrightarrow{a}$ in H gets sent to a number in $[0,180]$ by f, we have

$$m\angle as = |f(\overrightarrow{a}) - f(\overrightarrow{s})| < 180. \tag{1}$$

In other words, since $f(\overrightarrow{s})$ is at neither extreme of the interval $[0,180]$, it cannot be 180 away from any other number in that interval.

On the other hand, consider the half-rotation $HR(\overrightarrow{s}, P)$. The lines $\overleftrightarrow{s}$ and $\overleftrightarrow{r}$ coincide because the two rays are opposite, so that the two half-rotations are the same too: $HR(\overrightarrow{s}, P) = H$. Now, applying Version 2 of the Protractor Postulate to the ray $\overrightarrow{s}$ and the point P, we get a bijection $g : H \to [0,180]$ with $g(\overrightarrow{s}) = 0$ and satisfying the second bullet point of Version 2. By the surjectivity, of g, we can find $\overrightarrow{a} \in H$ such that $g(\overrightarrow{a}) = 180$. But now we have

$$m\angle as = |g(\overrightarrow{a}) - g(\overrightarrow{s})| = |180 - 0| = 180.$$

This contradicts (1), which was supposed to hold for *all* $\overrightarrow{a} \in H$.

The contradiction must stem from our assumption that $f(\overrightarrow{s}) \neq 180$, so we are done. ■

In other words, the missing condition from Version 1 is still there in Version 2, but secretly.

Incidentally, let me also note that this is exactly the sort of thing you are supposed to do when studying axiomatic systems: try to trim them down, throw away or simplify axioms, without giving up any of the strength of the system. There are several reasons you might want to do this:

- If you are studying axiomatic systems anyway, it might well be that this is intrinsically interesting to you. Cutting out a redundant piece of an axiom (or an entire axiom) without giving up any of the power of the axiomatic system will force you to better understand how the individual components of the system come together.
- There is also a more practical aspect: Suppose you come up with some interpretation and want to decide whether it is actually a model of your axiomatic system. This means checking whether or not the axioms hold in your interpretation, and the fewer axioms you have and the simpler these are, the easier it will be to perform this check.

In its own small way, the analysis from above is of the same type as the big problem that I mentioned kept geometers occupied for two thousand years or so: figuring out whether or not Euclid's Postulate 5 is redundant. The above is massively less important and easier, but it's in the same spirit.

So to push this further, I'll leave you with a natural question. Now that we're hacking away at Axiom 8, we might as well try to drop *both* the first and the second bullet points from Version 1:

Version 3. *For every ray $\overrightarrow{r}$ and every point P not on the line $\overleftrightarrow{r}$ there is a bijection $f : HR(\overrightarrow{r}, P) \to [0,180]$ such that the following condition holds:*

- *For every two rays $\overrightarrow{a}, \overrightarrow{b} \in HR(\overrightarrow{r}, P)$ we have $m\angle ab = |f(\overrightarrow{a}) - f(\overrightarrow{b})|$.*

You might then ask whether the axiomatic system obtained by adjoining Version 3 to NG axioms (1) - (7) is as powerful as PNG1 (which by Theorem 1 is as strong as PNG2). I don't know the answer..