Math 340 A
Spring 2016

Wrapping up the April 11 lecture

This note is meant to supplement the lecture for Monday, April 11, where I left a claim unproven. There are plenty of these floating around in the lectures, so I just wanted to wrap this one up nicely.

The setup was as follows:

We had finite-dimensional subspaces U and W of a vector space V, and we fixed direct sum decompositions

$$U = (U \cap W) \oplus U' \text{ and } W = (U \cap W) \oplus W'. \tag{1}$$

Then, the claim was:

Lemma 1. *Under the above conditions, we have the direct sum decomposition*

$$U + W = U' \oplus (U \cap W) \oplus W'. \tag{2}$$

Proof. There are two things I need to prove here:

Part 1: The sum on the right hand side of (2) equals $U + W$. Indeed, the first two summands recover U as $U' \oplus (U \cap W)$ (by (1)) and similarly, the second and third summands give me back $W = (U \cap W) \oplus W'$. But this means that both U and W are contained in the right hand side of (2), and hence so is $U + W$ (because the latter is the smallest subspace of V containing both U and W; see 1.39 on page 20 of our textbook).

Part 2: The sum on the right hand side of (2) is direct. Here, I'll use the characterization 1.44 on page 23 of your book for a sum to be direct:

I need to prove that if I choose

$$u' \in U',\ v \in U \cap W,\ w' \in W'$$

such that $u' + v + w' = 0$, then all u', v and w' must vanish (i.e. they're all 0).

Now, $u' + v + w' = 0$ can be rewritten as

$$u' = -(v + w'),$$

which belongs to W because both v and w' do. So $u' \in U' \subseteq U$ as well as $u' \in W$, and hence $u' \in U \cap W$. But remember that I chose $u' \in U'$, and the sum $U' \oplus (U \cap W)$ is direct! This means that $u' = 0$.

Similarly (just reverse the roles of U and W) we have $w' = 0$, and hence

$$v = u' + v + w' = 0$$

as well. ■