Problem 4. The transform associated with a random variable $Y$ has the form

$$M_Y(s) = a^6(0.1 + 2e^s + 0.1e^{4s} + 0.4e^{7s})^6.$$ 

Find $a$, $p_Y(41)$, $p_Y(11)$, the third largest possible value of $Y$, and its corresponding probability.

Solution

Problem 4.
Let $X$ be a random variable associated with the transform

$$M_X(s) = a(0.1 + 2e^s + 0.1e^{4s} + 0.4e^{7s}).$$

Since $M_Y(s) = (M_X(s))^6$, $Y$ can be interpreted as the sum of six independent, identically distributed random variables $X_1, \ldots, X_6$ associated with the transform $M_X(s)$. Using the fact $M_X(0) = 1$, we obtain $a = 1/(0.1 + 2 + 0.1 + 0.4) = 5/13$.

Each of $X_i$ takes values in the set $\{0, 1, 4, 7\}$. There is no combination of six values from that set that yields a sum of 41 and, therefore, $p_Y(41) = 0$.

We have $Y = 11$ for the following combinations of values for the $X_i$'s:

(i) $(\binom{6}{4}) \cdot (\binom{2}{1})$ ways to have four 0's, one 4, and one 7

(ii) $(\binom{6}{5}) \cdot (\binom{3}{2})$ ways to have one 0, four 1's, and one 7

(iii) $(\binom{6}{3}) \cdot (\binom{3}{2})$ ways to have one 0, three 1's, and two 4's.

As a result,

$$p_Y(11) = \binom{6}{4} \binom{2}{1} \left( \frac{5}{13} \right)^6 (0.1)^4(0.1)(0.4) + \binom{6}{5} \binom{2}{1} \left( \frac{5}{13} \right)^6 (0.1)(2)^4(0.4)$$

$$+ \binom{6}{3} \binom{3}{2} \left( \frac{5}{13} \right)^6 (0.1)(2)^3(0.1)^2$$

$$= 0.0637.$$ 

The third largest possible value of $Y$ is 36. It occurs when four of the $X_i$ are equal to 7 and the other two are equal to 4, or when five of the $X_i$ are equal to 7 and the other one is equal to 1. We have

$$p_Y(36) = \binom{6}{4} \left( \frac{5}{13} \right)^6 (0.4)^4(0.1)^2 + \binom{6}{5} \left( \frac{5}{13} \right)^6 (0.4)^5 \cdot 2 = 4 \cdot 10^{-4}.$$
Problem 6. The transform and the mean associated with a discrete random variable $X$ are given by
\[ M(s) = ae^s + be^{4(e^s - 1)}, \quad E[X] = 3. \]

Find:
(a) The scalar parameters $a$ and $b$.

Solution

Problem 6.
(a) Since $M(0) = 1$, we have
\[ a + b = 1. \]

Since
\[ E[X] = \left. \frac{d}{ds} M(s) \right|_{s=0} = (a + 4be^{s(e^s - 1)}) \bigg|_{s=0} = 3, \]
we obtain
\[ a + 4b = 3. \]

Solving the two equations for $a$ and $b$, we have
\[ a = \frac{1}{3}, \quad b = \frac{2}{3}. \]

Problem 11. Mean and variance of the Poisson. Use the formula for the transform associated with a Poisson random variable $X$ to calculate $E[X]$ and $E[X^2]$.

Problem 11.
Let $\lambda$ be the parameter of the Poisson random variable $X$. The corresponding transform is given by
\[ M_X(s) = e^{\lambda(e^s - 1)}, \quad s < \lambda. \]

Using the formula
\[ \left. \frac{d^n}{ds^n} M(s) \right|_{s=0} = E[X^n], \]
we obtain
\[ E[X] = \left. \lambda e^s e^{\lambda(e^s - 1)} \right|_{s=0} = \lambda \cdot 1 \cdot e^{\lambda(1-1)} = \lambda, \]
\[ E[X^2] = \left( \lambda e^s e^{\lambda(e^s - 1)} + e^s \cdot \lambda e^s \cdot e^{\lambda(e^s - 1)} \right) \bigg|_{s=0} = \lambda + \lambda^2. \]
Problem 15. Let $X_1$ and $X_2$ be independent random variables with the same PMF:

$$p_{X_1}(x) = p_{X_2}(x) = \begin{cases} 
1/4, & \text{if } x = 1, \\
1/4, & \text{if } x = 2, \\
1/2, & \text{if } x = 3, \\
0, & \text{otherwise.}
\end{cases}$$

Use convolution to obtain the PMF of $Y = X_1 + X_2$.

Problem 14.
Using either the convolution formula, or the graphical method, we obtain

$$p_Y(y) = \begin{cases} 
1/16, & \text{if } y = 2, \\
2/16, & \text{if } y = 3, \\
5/16, & \text{if } y = 4, \\
4/16, & \text{if } y = 5, \\
4/16, & \text{if } y = 6, \\
0, & \text{otherwise.}
\end{cases}$$

Problem 24. The random variables $X$ and $Y$ are described by a joint PDF which is constant within the unit area quadrilateral with vertices $(0,0)$, $(0,1)$, $(1,2)$, and $(1,1)$. Use the law of total variance to find the variance of $X + Y$.

Problem 23.
We will condition on $X$ and use the law of total variance

$$\text{var}(X + Y) = \mathbb{E}\left[\text{var}(X + Y \mid X)\right] + \text{var}\left(\mathbb{E}[X + Y \mid X]\right).$$

Given a value $x$ of $X$, the random variable $Y$ is uniformly distributed in the interval $[x, x + 1]$, and the random variable $X + Y$ is uniformly distributed in the interval $[2x, 2x + 1]$. Therefore, $\mathbb{E}[X + Y \mid X] = 0.5 + 2X$ and $\text{var}(X + Y \mid X) = 1/12$. Thus,

$$\text{var}(X + Y) = \text{var}(0.5 + 2X) + \text{var}(1/12) = 4\text{var}(X) + \text{var}(1/12) = \frac{5}{12}.$$
Problem 26.

(a) You roll a fair six-sided die, and then you flip a fair coin the number of times shown by the die. Find the expected value and the variance of the number of heads obtained.

(b) Repeat part (a) for the case where you roll two dice, instead of one.

Problem 25.

(a) Let $X_i$ be independent Bernoulli random variables that are equal to 1 if the $i$th flip results in heads. Let $N$ be the number of coin flips. We have $E[X_i] = 1/2$, $\text{var}(X_i) = 1/4$, $E[N] = 7/2$, and $\text{var}(N) = 35/12$. (The last equality is obtained from the formula for the variance of a discrete uniform random variable.) Therefore, the expected number of heads is

$$E[X_i]E[N] = \frac{7}{4},$$

and the variance is

$$\text{var}(X_i)E[N] + E[X_i]^2\text{var}(N) = \frac{1}{4} \cdot \frac{7}{2} + \frac{1}{4} \cdot 3512 = \frac{77}{48}.$$

(b) The experiment in part (b) can be viewed as consisting of two independent repetitions of the experiment in part (a). Thus, both the mean and the variance are doubled and become $7/2$ and $77/24$, respectively.

Problem 33. Consider $n$ independent tosses of a die. Each toss has probability $p_i$ of resulting in $i$. Let $X_i$ be the number of tosses that result in $i$. Show that $X_1$ and $X_2$ are negatively correlated (i.e., a large number of ones suggests a smaller number of twos).

Problem 33.

Let $A_i$ (respectively, $B_i$) be a Bernoulli random variable which is equal to 1 if and only if the $i$th toss resulted in 1 (respectively, 2). We have $E[A_i B_i] = 0$ and $E[A_i B_s] = E[A_i]E[B_s] = p_1 p_2$ for $s \neq t$. We have

$$E[X_1 X_2] = E[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)] = n E[A_1(B_1 + \cdots + B_n)] = n(n-1)p_1 p_2,$$

and

$$\text{cov}(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2] = n(n-1)p_1 p_2 - np_1 np_2 = -np_1 p_2.$$
Problem 34. Let $X = Y - Z$ where $Y$ and $Z$ are nonnegative random variables such that $YZ = 0$.

(a) Show that $\text{cov}(Y, Z) \leq 0$.
(b) Show that $\text{var}(X) \geq \text{var}(Y) + \text{var}(Z)$.
(c) Use the result of part (b) to show that

$$\text{var}(X) \geq \text{var}\left(\max\{0, X\}\right) + \text{var}\left(\max\{0, -X\}\right).$$

Problem 34.

(a) Let $m_Y$ and $m_Z$ be the means of $Y$ and $Z$, respectively. Note that these means are nonnegative. We have

$$\text{cov}(Y, Z) = \mathbb{E}[YZ] - m_Y m_Z = -m_Y m_Z \leq 0.$$

(b) We have

$$\text{var}(X) = \text{var}(Y) + \text{var}(Z) - 2\text{cov}(Y, Z) \geq \text{var}(Y) + \text{var}(Z).$$

(c) Let $Y = \max\{0, X\} \geq 0$ and $Z = \max\{0, -X\} \geq 0$. Note that $X = Y - Z$ and $YZ = 0$. The result follows from part (b).