Why do we need utility in addition to expected values?

Preference Relations and Utilities

Common Utility Functions

Problems with Utility Theory
Why do we need utility in addition to expected values?

Preference Relations and Utilities
- Preference Relations
- Rationality Assumption
- Utility Function

Common Utility Functions
- Common 1-D Utility Functions
- Common 2-D Utility Functions

Problems with Utility Theory
Example A

Consider the decision:

- Option A: $1 vs. Option B: $10 w.p. 0.4
- $0 w.p. 0.6

Using expected values, the choice is simple: select B. Now ask yourself what you would do if the numbers are multiplied by a million... Option A': $1M vs. Option B': $10M w.p. 0.4
- $0 w.p. 0.6

Your answer might change... WHY??
Example A

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- Your answer might change... WHY??
Risk Attitudes

Three possible risk attitudes:

- Risk averse
- Risk neutral
- Risk seeking
Example B

- What’s your favorite coffee/drink/food?
Example B

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- How about increase your normal daily amount by 1000 times?
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GOODS, BADS and NEUTRALS

Utility

Units of coffee are goods
Units of coffee are bads

Around x' units, a little extra coffee is a neutral.
In the preference-based approach, the objectives of the decision maker are summarized in a *preference relation*.

- It is an individual property!!!
- We denote by $\succeq$: a binary relation on the set of alternatives $x, y \in X$
- We read $x \succeq y$ as “$x$ is at least as good as $y$”
Two important relations on $X$

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   $$x > y \iff \quad$$
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   read "$x$ is preferred to $y$"

2. The *indifference* relation, $\sim$, defined by

   \[ x \sim y \iff \]

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Those assumptions could be difficult to satisfy when dealing with alternatives far from common experience.
Rationality Assumption

The preference relation $\succeq$ is rational if it possess the following two properties:

1. **Completeness**: for all $x, y \in X$, we have that $x \succeq y$ or $y \succeq x$ (or both)
Rationality Assumption

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1. **Completeness**: for all $x, y \in X$, we have that $x \succeq y$ or $y \succeq x$ (or both)

2. **Transitivity**: For all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$. 
Preference Relations

Rationality Assumption

Utility Function

Proposition: If $\succsim$ is rational then...

If $\succsim$ is rational then:

- $\succ$ is both **irreflective** ($x \succ x$ never holds) and **transitive** (if $x \succ y$ and $y \succ z$, then $x \succ z$).
- $\sim$ is reflective ($x \sim x$ for all $x$), transitive (if $x \sim y$ and $y \sim z$, then $x \sim z$), and symmetric (if $x \sim y$, then $y \sim x$).
- If $x \succ y \succsim z$, then $x \succ z$.

**Proof:** part of Homework 1; using definition and rationality assumptions.
We describe preference relations by means of a utility function.

- A utility function $u(x)$ assigns a numerical value to each element in $X$, ranking the elements of $X$ in accordance with the individual’s preferences. Officially,
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**Definition**

A function $u : X \to \mathbb{R}$ is a *utility function representing preference relation* $\succeq$ if, for all $x, y \in X$,

$$x \succeq y \iff u(x) \geq u(y)$$
Note that a utility function represents a preference relation \( \succsim \) is not unique.
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For any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, $v(x) = f(u(x))$ is a new utility function representing the same preferences as $u(\cdot)$. (part of Homework 1) E.g., $2u(x)$

Therefore, only the ranking of alternatives that matters.

E.g., if $u(x) = 6$ and $u(y) = 2$, then $x \succ y$; but $x$ is not necessarily “three times better than” $y$. 
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Common 1-D Utility Functions

\[ u(x) = a + b \times x \] (1)

\[ u(x) = \sqrt{x} \] (2)

\[ u(x) = \log(x) \] (3)

\[ u(x) = x^\beta \] (4)

\[ u(x) = \exp(x) \] (5)

risk seeking, neutral, and averse? Calculate the insurance premium?
Common 2-D Utility Functions

\[ X \text{ might be two or multi-dimentional; e.g., } X = \mathbb{R}^n \]

- **Cobb-Douglas Utility Function**  
  \[ u(x_1, x_2) = x_1^a x_2^b (a > 0, b > 0) \]

- **Perfect Substitutes Utility Function** (Linear)  
  \[ u(x_1, x_2) = ax_1 + bx_2 \]

- **Perfect Complements Utility Function**  
  \[ u(x_1, x_2) = \min(x_1, x_2) \]
Common 2-D Utility Functions

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Then we can draw the indifference curves (to compare with budget, applied in LP?), calculate marginal utilities (and compare with each other, as well as marginal costs, etc.)
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**Indifference Curves for Cobb-Douglas Utility Functions**

**COBB DOUGLAS INDIFFERENCE CURVES**

All curves are “hyperbolic”, asymptoting to, but never touching any axis.
Indifference Curves for Perfect Substitutes Utility Function

PERFECT SUBSTITUTES

\[ V(x_1, x_2) = x_1 + x_2 \]

All are linear and parallel.
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Indifference Curves for Perfect Complements Utility Function

PERFECT COMPLEMENTS

\[ W(x_1, x_2) = \min\{x_1, x_2\} \]

All are right-angled with vertices/corners on a ray from the origin.
Problems with Utility Theory

- Entire risk profile cannot be captured with a single number (expected utility)
- Utility has no meaning to most people
- No natural utility function (i.e., when should we use log or square root?)
- People violate axioms