(P_{uk}, d_k) \text{ as } \ldots \ldots .

Bazaraa et al. (1978) show that one of the above stopping criteria is met after finitely many steps. Since each norm requires a somewhat different and technical proof we refer the reader to the above source. The main idea of each proof is to show that each combination of solutions to (P_{uk}), i.e. \( x \in u(P_{uk}) \), and zero components of \( u^k \) can only occur one time.

18.6. LAGRANGIAN DUALITY AND BRANCH AND BOUND

The method of Lagrangian duality may be used together with branch and bound and implicit enumeration methods. There are many degrees of freedom and many variations, but the idea is to use Lagrangian duality generally in place of linear programming to help fathom the solutions. One particular Lagrangian relaxation of the problem is selected for use for the solution of the entire problem. The Lagrangian dual problem is not generally completely solved at each step; only a bound need be computed. See the flow chart in Figure 18.6. The main difference between this and ordinary branch and bound is that here the Lagrangian relaxation replaces linear programming relaxation.

The Algorithm

We begin with the original problem and a Lagrangian relaxation (step 1). We next select a set of weights and obtain a bound using the Lagrangian (steps 2 and 3). If the bound is sufficiently large as to fathom the solution (step 4), we do so. Otherwise, we check to see whether the current solution is
Figure 18.6

Flow Chart of Lagrangian Branch-and-Bound Procedure
feasible with respect to the relaxation constraints (step 5). If so, we check to see if it is an improvement over the previous incumbent (step 6), and then check to see whether the complementary slackness conditions \( \{u(b-Ax) = 0\} \) are satisfied (step 7). If the complementary slackness conditions are satisfied, the branch is fathomed (step 8), and we check to see whether there are any more solutions on the list to examine (step 9). If there are no solutions on the list, we stop (step 10); otherwise we choose a solution on which to branch (step 11), and go to step 2. If (in step 7) the complementary slackness conditions are not satisfied, or if the Lagrangian solution is not feasible in step 5, we decide whether or not to try to find an improved set of weights \( u \) for the Lagrangian (step 12). If so, we choose a new set of multipliers (step 13) and go to step 3. Otherwise, we branch on a variable of the solution, and add the corresponding subproblems to the list of problems (step 14), and go to step 11.

A few points are in order. When we proceed from one node to a successor node, it is advantageous to use the Lagrange multipliers of the predecessor node as a starting set of multipliers for the successor node. Observe this in the example where the initial set of multipliers frequently turns out to be optimal. Second, we generally find that few iterations are needed to solve problems as we proceed down the tree. We shall illustrate the method using the problem of Example 18.8. In our illustration we shall revise the \( u \)'s to their optimal values by examining all possible solutions. In general, one of the methods
described previously is used to revise the u's. This change is for convenience, and to illustrate some of the ramifications of the procedure.

EXAMPLE 18.11:

Our problem is to

Minimize \[ 5x_1 + 7x_2 + 10x_3 + 3x_4 + x_5 \]
subject to: \[ x_1 - 3x_2 + 5x_3 + x_4 - 4x_5 \geq 2 \]
\[ -2x_1 + 6x_2 - 3x_3 - 2x_4 + 2x_5 \geq 0 \]
\[ -x_2 + 2x_3 - x_4 - x_5 \geq 1 \]
\[ x_1, \ldots, x_5 = 0 \text{ or } 1. \]

By using a Lagrangian relaxation on the second constraint, we obtain

Minimize \[ 5x_1 + 7x_2 + 10x_3 + 3x_4 + x_5 \]
\[ + u_2(0 + 2x_1 - 6x_2 + 3x_3 + 2x_4 - 2x_5) \]
subject to: \[ x_1 - 3x_2 + 5x_3 + x_4 - 4x_5 \geq 2 \]
\[ - x_2 + 2x_3 - x_4 - x_5 \geq 1 \]
\[ x_1, \ldots, x_5 = 0 \text{ or } 1. \]

Feasible solutions to this relaxation are as follows:
<table>
<thead>
<tr>
<th>Solution</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>Objective Function Value</th>
<th>Value of Lagrangian</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>17</td>
<td>17-3u_2</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>13</td>
<td>13+5u_2</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>22</td>
<td>22- u_2</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>18</td>
<td>18+7u_2</td>
</tr>
<tr>
<td>F</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>16</td>
<td>16+3u_2</td>
</tr>
<tr>
<td>G</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>10+3u_2</td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>15+5u_2</td>
</tr>
</tbody>
</table>

(only solutions A and D are feasible to the original problem)

Refer to Figures 18.7 throughout this discussion. By maximizing the minimum value of the Lagrangian over all solutions to the relaxation, we find solution 1 in the Figure to be optimal for the Lagrangian dual. From the solution we obtain a bound of 13.5 and a value of $u_2 = 7/6$ (as per Example 18.6). This value is achieved for solutions A and G: $(17-(7/6)(3)); (10+(7/6)(3))$. In general only one solution is identified by the solution procedure; accordingly assume we find solution G. Since solution G does not satisfy the relaxed constraint, we do not obtain an incumbent solution. (If we had found solution A which satisfies the relaxed constraint we would, of course, have obtained an incumbent solution.) We arbitrarily choose $x_1$ as the branching variable. First consider the branch $x_1 = 0$. That branch admits three solutions: A, B, and G. We begin with the multipliers for the predecessor solution: (in this case $u_2 = 7/6$) and find
<table>
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<th>Objective Function Value</th>
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</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>17</td>
<td>17-3u_2</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>13</td>
<td>13+5u_2</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>22</td>
<td>22- u_2</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>18</td>
<td>18+7u_2</td>
</tr>
<tr>
<td>F</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>16</td>
<td>16+3u_2</td>
</tr>
<tr>
<td>G</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>10+3u_2</td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>15</td>
<td>15+5u_2</td>
</tr>
</tbody>
</table>

Refer to Figure 18.7 throughout this discussion. By maximizing the minimum value of the Lagrangian over all solutions to the relaxation, we find solution 1 in the figure to be optimal for the Lagrangian dual. From the solution we obtain a bound of 13.5 and a value of $u_2 = 7/6$ (as per Example 18.6). This value is achieved for solutions A and G: $(17-(7/6)(3))$; $(10+(7/6)(3))$. In general only one solution is identified by the solution procedure; accordingly assume we find solution G. Since solution G does not satisfy the relaxed constraint, we do not obtain an incumbent solution. (If we had found solution A which satisfies the relaxed constraint we would, of course, have obtained an incumbent solution.) We arbitrarily choose $x_1$ as the branching variable. First consider the branch $x_1 = 0$. That branch admits three solutions: A, B, and G. We begin with the multipliers for the predecessor solution: (in this case $u_2 = 7/6$) and find
Figure 18.7
that the solution optimal at that node as well as the value of the multiplier remain the same. Accordingly, solution 2 is the same as solution 1.

The branch \( x_1 = 1 \) restricts us to solutions D, E, F, and H. Beginning with \( u_2 = 7/6 \), we have a bound of

For D: \( 22 - (7/6)(1) = 125/6 \)
For E: \( 18 + (7/6)(7) = 157/6 \)
For F: \( 16 + (7/6)(3) = 39/2 = 117/6 \)
For H: \( 15 + (7/6)(5) = 125/6 \)

By reference to Figure 18.7, we see that this solution is not optimal. The minimum upper bound is that of F. If we increase \( u_2 \), the bound for D decreases, and if we increase \( u_2 \), the bounds for E, F, and H all increase. We increase \( u_2 \) to 3/2 at which time any further increase will decrease the minimum upper bound. We may find either solution D or F, each having a bound of 20.5. We assume that we find solution D. Because solution D also satisfies the relaxed constraint, it is feasible. We therefore now have an incumbent solution that has a value of 22.

We next choose solution two as our solution for branching because it has the smaller bound. Next, choose \( x_2 \) as our branching variable. (Here our choice is arbitrary.) The branch \( x_2 = 0 \) restricts us to solutions B, and G. Beginning with \( u_2 = 3/2 \) we have the following bounds:

For B: \( 13 + 5u_2 \)

For G: \( 10 + 3u_2 \)
The minimum for \( u_2 = 3/2 \) is \( G(4.5) \), but we may increase \( u_2 \) and increase the bound. We can increase \( u_2 \) without limit; that means there is an infinite bound. When the Lagrangian dual has an infinite value, the corresponding primal problem has no feasible solution. Therefore, we fathom this branch.

Next we branch on \( x_2 = 1 \). The result is solution 5. The only feasible solution is solution A with the following bound:

For A: 17-3\( u_2 \)

The initial solution (\( u_2 = 3/2 \)) is A with \( z = 12.5 \). By decreasing \( u_2 \) to zero, we can increase the bound to 17. Because solution A is feasible, we now have an incumbent solution A with \( z = 17 \). The optimal value of \( u_2 = 0 \); consequently the complementary slackness conditions hold and we may fathom this branch.

We have one active solution remaining, solution 3. Because the bound for that solution of 20.5 is greater than that of our current incumbent, we may now fathom solution 3. The procedure is complete; solution A is optimal. See the tree of solutions in Figure 18.8.

18.6.3 OTHER ASPECTS OF LAGRANGIAN BRANCH AND BOUND

Various aspects of the branch and bound approach should be emphasized. For example in step 4 of Figure 18.5 the value of the Lagrangian relaxation \( u(P_u) \) gives a valid lower bound on the Lagrangian dual and of course the original problem. If that lower bound is greater than the corresponding value of
Figure 18.8
Tree of Lagrangian Branch and Bound Solution
Figure 18.8
Tree of Lagrangian Branch and Bound Solution
the incumbent, the branch may be fathomed. Similarly, we may
decide to stop trying to improve the dual (step 12) when our
progress in increasing the value of the Lagrangian relaxation
slows sufficiently. In such cases we do not generate a bound
that allows us to fathom, but we may not be able to generate such
a bound for that subproblem in any event.

We may be able to use Lagrange multipliers to help choose
branching variables. If the constraints that are relaxed are
separable (each variable appears in only one relaxed constraint),
we may choose a constraint for which the adjoined constraint
value \( u_i(b_i - \sum_j a_{ij}x_j) \) is maximum and choose a variable that
appears in that constraint. This rule tends to be particularly
effective with generalized upper bound constraints. The reason
is that by branching on a high cost variable, we are likely to
fathom one node. This type of structure is found in the
Generalized Assignment Problem, and in other problems involving
multiple choice constraints.

We may also use conditional bounds based on problem
structure to help us in the choice of branching variable, as well
as in restricting the values of variables. We shall illustrate
by means of two simple examples.

EXAMPLE 18.12: Consider the following relaxation of a
binary integer programming problem:

\[
\text{Minimize} \quad \sum_{j=1}^{n} c_j x_j + u(b-Ax) \quad \text{or}
\]

\[
(P_u) \quad \text{Minimize} \quad \sum_{j=1}^{n} (c_j - ua_j) x_j + ub
\]

subject to: \( x_j = 0 \) or \( 1 \), \( j=1, \ldots, n \)
Clearly a solution to this relaxation is \( x_j = 1 \) if \( c_j - u_{aj} < 0 \) and \( x_j = 0 \) if \( c_j - u_{aj} \geq 0 \). We may draw two conclusions as to the results of branching on \( x_j \) (where \( x_j \) is a free variable):

1. Where \( x_j \) is one in an optimal solution of \( (P_u) \), the branch \( x_j = 1 \) gives the same value of the Lagrangian relaxation, i.e., \( \nu(P_u) \). The branch \( x_j = 0 \) gives a value of the Lagrangian relaxation of \( \nu(P_u) - (c_j - u_{aj}) \).

2. Where \( x_j \) is zero in an optimal solution of \( (P_u) \), the branch \( x_j = 0 \) gives the same value of the Lagrangian relaxation, i.e. \( \nu(P_u) \). The branch \( x_j = 1 \) gives a value of the Lagrangian relaxation of \( \nu(P_u) + (c_j - u_{aj}) \).

**EXAMPLE 18.13:** Consider the following relaxation of a binary integer programming problem with separable multiple choice constraints:

\[
\begin{align*}
\text{Minimize} & \quad \sum_j (c_j - u_{aj})x_j + ub \\
\text{subject to:} & \quad \sum_{j \in N_i} x_j = 1 \quad i=1, \ldots, m \\
& \quad x_j = 0 \text{ or } 1, \quad j=1, \ldots, n
\end{align*}
\]

Clearly a solution to this relaxation is \( x_j = 1 \) if \( c_j - u_{aj} = \min \{c_h - u_{ah}\} \), and \( x_j = 0 \) otherwise. As in Example 18.12, we may draw two conclusions as to the results of branching on \( x_j \) (a free variable):

1. Where \( x_j \) is one in an optimal solution of \( P_u \), the branch \( x_j = 1 \) gives the same value of the Lagrangian.
relaxation, i.e., \((P_u)\). The branch \(x_j = 0\) gives the value of the Lagrangian relaxation of \((P_u) - (c_j - ua^j) + (c_k - ua^k)\) where \(k\) is such that \(c_k - ua^k = \min_{h \in N_i} \{c_h - ua^h\}\).

2. Where \(x_j\) is zero in an optimal solution of \(P_u\), the branch \(x_j = 0\) gives the same value of the Lagrangian relaxation, i.e., \((P_u)\). The branch \(x_j = 1\) gives the value of the Lagrangian relaxation of \((P_u) + (c_j - ua^j) - \min_{h \in N_i} \{c_h - ua^h\}\).

The conditional bounds obtained depend upon the structure of the problem, but once they are established they may be used as discussed in Section 15.3.

Another interesting point to be emphasized is that any solution to the Lagrangian relaxation should be tested to see whether or not it satisfies the relaxed constraints. If the value \(u(b - Ax) > 0\), the corresponding solution \(x\) cannot be feasible to the original problem. If the value \(u(b - Ax) \leq 0\), the corresponding solution \(x\) may be feasible to the original problem, and may be a potential incumbent. It is necessary to test \(b - Ax \leq 0\) to verify this.

Although we would like to be able to solve the Lagrangian relaxation or the Lagrangian dual quickly and efficiently, our real aim is solve the integer problem. To do so, a subgoal is to derive effective bounds, and if possible find relaxations and multipliers that yield good if not optimal solutions. It is particularly helpful if the Lagrange multipliers yield good feasible solutions to the original problem \((P)\), solutions that we may use as incumbents to help fathom branches.