MTH306 Notes – Jun 6

Section 3.1-3.3 Homogeneous linear equations

- Homogeneous linear equation:

\[ y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x) = 0. \]

Initial conditions:

\[ y(a) = b_0, \ y'(a) = b_1, \ldots, y^{(n-1)}(a) = b_{n-1}. \]

- Theorem: If \( y_1, \ldots, y_n \) are \( n \) linearly independent solutions of the homogeneous linear equation:

\[ y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x) = 0. \]

Then the general solution is

\[ Y(x) = c_1y_1(x) + \cdots + c_ny_n(x). \]

- Question 1: What is linear independence?

Answer: If \( y_1(x), \ldots, y_n(x) \) are given, and define the Wronskian by

\[
W = \begin{vmatrix}
    y_1(x) & y_2(x) & \cdots & y_n(x) \\
    y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\
    \vdots & \vdots & \ddots & \vdots \\
    y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x)
\end{vmatrix}
\]

- \( y_1, \ldots, y_n \) are linearly dependent \( \iff \) \( W \equiv 0 \).
- \( y_1, \ldots, y_n \) are linearly independent \( \iff \) \( W \neq 0 \) for some \( x \).

Case \( n = 2 \)

\[ W = \begin{vmatrix}
    y_1(x) & y_2(x) \\
    y_1'(x) & y_2'(x)
\end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x) \]

If \( W \neq 0 \) for some \( x \), then \( y_1, y_2 \) are linearly independent.

Example. §3.1 # 25 Determine whether the pair of functions are linearly independent of dependent on the real line.

\[ f(x) = e^x \sin x, \quad g(x) = e^x \cos x. \]
Solution: Derivatives (product rule): \( f'(x) = e^x \sin x + e^x \cos x \), \( g'(x) = e^x \cos x - e^x \sin x \).

Wronskian:
\[
W = \begin{vmatrix}
  f(x) & g(x) \\
  f'(x) & g'(x)
\end{vmatrix} = f(x)g'(x) - f'(x)g(x) \\
= e^x \sin x \cdot (e^x \cos x - e^x \sin x) - e^x \cos x \cdot (e^x \sin x + e^x \cos x) \\
= -(e^x)^2(\sin^2 x + \cos^2 x) \\
= -e^{2x} \neq 0
\]

Since Wronskian of \( f \) and \( g \) is not zero, we obtain that \( f \) and \( g \) are linearly independent.

• Question 2: How to find \( n \) independent solutions?

Answer for constant coefficient case: Use characteristic equation.

• Constant coefficient homogeneous linear equation:
\[
a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0, \quad (a_n \neq 0)
\]

Characteristic equation: \( y^{(n)} \to r^n, \cdots, y'' \to r^2, \ y' \to r, \ y \to 1. \)
\[
a_n r^n + a_{n-1} r^{(n-1)} + a_{n-2} r^{(n-2)} + \cdots + a_1 r + a_0 = 0.
\]

Example. Find the characteristic equation.

(1) \( 4y'' + 7y = 0. \) Solution: \( 4r^2 + 7 = 0. \)

(2) \( 5y^{(4)} + y^{(3)} - 3y' - y = 0. \) Solution: \( 5r^4 + r^3 - 3r - 1 = 0. \)

• Roots of characteristic equation. If \( r \) is a root of the characteristic equation:

<table>
<thead>
<tr>
<th>( r ) is a distinct root</th>
<th>Multiplicity is 1</th>
<th>( e^{rx} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r ) is a repeated root</td>
<td>Multiplicity is ( k )</td>
<td>( e^{rx}, xe^{rx}, \cdots, x^{k-1}e^{rx} )</td>
</tr>
<tr>
<td>( r ) is a complex root</td>
<td>( r = a \pm b i ) (pair)</td>
<td>( e^{ax} \cos bx, \ e^{ax} \sin bx )</td>
</tr>
</tbody>
</table>

– Factor the characteristic equation to see the multiplicity.
(In Maple, use command \texttt{factor(expression)}. )
Example. The characteristic equation: \( r^5 + r^3 - 12r + 2r^2 + 8 = 0 \).
Solution: Factor the characteristic equation: \((r + 2) \cdot (r - 1)^2 \cdot (r^2 + 4) = 0\). Consider the roots:

<table>
<thead>
<tr>
<th>( r_1 = -2 )</th>
<th>a distinct root</th>
<th>Multiplicity is 1</th>
<th>( e^{-2x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_2 = 1 )</td>
<td>a repeated root</td>
<td>Multiplicity is 2</td>
<td>( e^x, xe^x )</td>
</tr>
<tr>
<td>( r_{3,4} = \pm 2i )</td>
<td>a pair of complex roots</td>
<td>( a = 0, b = 2 )</td>
<td>( \cos 2x, \sin 2x )</td>
</tr>
</tbody>
</table>

General solution: \( Y(x) = c_1 e^{-2x} + c_2 e^x + c_3 xe^x + c_4 \cos 2x + c_5 \sin 2x \).

Example. §3.3 # 9 Solve \( 4y'' + 12y' + 9y = 0; \quad y(0) = 1, y'(0) = 2 \).
Solution: Characteristic equation. \( 4r^2 + 12r + 9 = 0 \). That is \((2r + 3)^2 = 0\). Roots:
\( r_1 = -\frac{3}{2} \) (multiplicity = 2); independent solutions: \( e^{-\frac{3}{2}x}, \quad xe^{-\frac{3}{2}x} \).
General solution:
\[
y(x) = c_1 e^{-\frac{3}{2}x} + c_2 xe^{-\frac{3}{2}x}
\]
Determine the constants: \( y(0) = c_1 e^0 + c_2 \cdot 0 = c_1 = 1 \). \( \Rightarrow c_1 = 1 \).
\( \Rightarrow y(x) = e^{-\frac{3}{2}x} + c_2 xe^{-\frac{3}{2}x} \).
\( \Rightarrow y'(x) = -\frac{3}{2} e^{-\frac{3}{2}x} + c_2 (e^{-\frac{3}{2}x} - \frac{3}{2} xe^{-\frac{3}{2}x}) \).
\( \Rightarrow y'(0) = -\frac{3}{2} e^{-\frac{3}{2} \cdot 0} + c_2 (e^{-\frac{3}{2} \cdot 0} - \frac{3}{2} \cdot 0 \cdot e^{-\frac{3}{2} \cdot 0}) = -\frac{3}{2} + c_2 = 2 \).
\( \Rightarrow c_2 = \frac{7}{2} \).
Particular solution: \( y(x) = e^{-\frac{3}{2}x} + \frac{7}{2} xe^{-\frac{3}{2}x} \).