Course Documents

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Lectures: Mon, Wed, Fri 10:00-10:50AM Talbert 107  
Recitations (Labs): C1: Wed 3:00 PM - 3:50 PM Talbert 106  
C2: Fri 3:00 PM - 3:50 PM Talbert 106  
TA: Andrew Samer  

Description: This is the second part of a 2-semester sequence in Calculus for students in social, biological, and management sciences. The course will cover introduction to functions of several variables, applications using both calculus and elementary linear programming techniques, elementary integration techniques, simple differential equations, and some probability theory.

Textbook: No mandatory textbook. Taking lecture notes is enough. If you want to read a textbook, you can simply get any standard calculus textbook. The official one for our course is L. Goldstein, D. Schneider, D. Lay, and N. Asmar, Calculus and Its Applications, 4th custom UB edition. (Chapters 7-12)

Prerequisites: MTH 121 with recommended grade of C or higher

Course website: UBLearns page. All course announcements, homework assignments, quiz information, practice exams and your grades will be posted on UBLearns.

Homework: Homework will be assigned on UBLearns in most weeks. Each homework set contains about 10 problems, and a few problems will be collected and graded. Final answer of all homework will be provided. Please turn in your homework in the recitation session.

Recitations (labs): Recitation will begin in the second week in your recitation classroom. In recitations, our TA will answer your questions related to homework and discuss important course topics. There will be 6 quizzes given in the recitation sessions. The dates are  
Session C1: 02/03, 02/10, 03/02, 03/30, 04/06, 04/27  
Session C2: 02/05, 02/12, 03/04, 04/01, 04/08, 04/29  
The lowest two quiz grade will be dropped. You will need a formal excuse to make up a quiz.

Exams: There will be four in-class exams. Each takes 50 minutes in the lecture room. There is no final exam. Exam dates are:  
Exam 1: 02/15 Exam 2: 03/11 Exam 3: 04/18 Exam 4: 05/06  
Please note that make-up exams will only be given in special circumstances. In particular, I must be notified beforehand if you require a make-up exam. Students may be asked to provide official documentation justifying the reason for missing an exam.
Grade components: Total 400 points.

Four Exams: 240 points (60 points per exam)
Quizzes: 80 points (20 points per quiz. Lowest 2 will be dropped.)
Homework: 80 points (10 points per assignment. Lowest 2 will be dropped)

Grade Scale:

A: 360 points or up, with an average 54 or higher in four exams.
A - : 345 points or up, with an average 48 or higher in four exams.
B+: 330 points or up.
B: 315 points or up.
B - : 300 points or up.
C+: 285 points or up.
C: 270 points or up.
C - : 250 points or up.
D: 220 points or up.
F: lower than 220

Ways to get bonus:

Presentation of topics: You are very welcome to give a 10-15 minute talk in the lecture, about any topic related to calculus, or even just related to math. The talk may focus on any application of calculus, history of calculus, interesting math discovery or unsolved problems. You can ask the instructor for a suitable topic or choose your own. As a reward, a bonus of up to 20 points will be added in the 400-point pool.

TA comments: During the semester the TA will report me any recommendation and important comments on students, especially the ones who don’t have solid math background but make great efforts and progress. So go to the recitation and join the discussion. You may be rewarded with up to 20 bonus points in the 400-point pool.

Use of Calculator: You are allowed to use any non-graphing scientific calculator in homework, quizzes and exams.

Math help center: There is a math help center in math building. Graduate student tutors available will assist you with math questions for free. The math help center starts on Feb. 8. The hours and location are Monday-Friday from 9am to 4pm at Math Building room 107.

Academic integrity: Students are expected to behave in accordance with the university policy on academic integrity. The guiding principle is that a student’s submitted work must be the student's own. Any kind of cheating will result in formal charges.

Additional information: If you have a diagnosed disability, please advise me during the first week of class so that we may review possible arrangements for reasonable accommodations.

Important dates:

02/01 Last day to add/drop
04/15 Last day to resign
03/14-03/18 No class due to Spring break.
Grading Guidelines

1. **Grading scale**: I don't suggest to use a point-by-point grading key. That would make grading very slow. Instead, the following grading scale will be used. You can judge students’ solutions and make a potential level grade (A, B, C, D, or F) in your mind and then give the corresponding (numbered) points on their homework, quiz and exam paper.

   - 10/10 - Correct method, clearly presented, correct answer. No mistakes (grade A work).
   - 7-9/10 - Correct method, clearly presented, almost correct answer. Displays general understanding of the concepts and methods. May contain one or more minor calculation mistakes, which don't essentially change the structure of the correct solution. (grade B/A- work)
   - 5-6/10 - Displays some understanding of the underlying concepts and ideas but the solution contains significant errors in execution of the details (grade C work).
   - 3-4/10 - Questionable understanding of the underlying concepts and ideas and/or major errors (grade D work).
   - 1-2/10 - Clear lack of understanding of the issues, but some parts of the solution are not totally incorrect (grade F with partial credit).
   - 0/10 - The solution has no redeeming features.

2. After grading quizzes or exams, please record the grade directly in the grade center on UB Learns page.

3. In grading, please clearly indicate where the mistake happens and write down the correct result for the mistake if that mistake can be easily corrected. For example, if the student writes \((x^2)' = 2x^2\), you should cross it out and write the correct rule \((x^2)' = 2x\).

4. Students are not required to simplify answers. For example, they don’t need to combine constant factors, simplify fractions over fractions in the denominator or write down \(e^0 = 1\) or \((x^2 + 1)^{-1} = \frac{1}{x^2 + 1}\) if unnecessary. What matters the most is if students can clearly show the understanding of the key ideas and methods in Calculus. For example, when they differentiate \((x^2 + 1)^2\), the answer from the chain rule \(2(x^2 + 1) \cdot 2\) is considered as a correct final answer. Students don’t need to multiply the two’s.

5. If a student makes a simple calculation mistake at the end of the solution, please consider to take off one or two points. If he makes the mistake at the beginning of the solution which doesn't essentially change the way of solving that problem, and he shows a totally correct method through the solution, you should consider to give at least 70% of points.

6. Students are required to show all key steps in the solution to show how they use the methods learned in calculus. A final answer without clear explanation doesn’t deserve more than 2 points out of 10.
Method of substitution for \( \int f(x) \, dx \):

1. Set up a function \( u \) of \( x \) and compute the differential \( du = u'(x) \, dx \).
2. Check if the integrand \( f(x) \, dx \) can be decomposed as:

\[
f(x) \, dx = (\text{a function of } u) \cdot (\text{a multiple of } du)
\]

3. Substitute \( x \)-terms by \( u \)-terms and form a new integral in terms of \( u \).
4. Evaluate the new integral to obtain the antiderivative in terms of \( u \).
5. Plug in the \( u \) expression to rewrite the antiderivative in terms of \( x \).

**Example** \( \int \frac{6x}{(x^2 + 1)^3} \, dx \).

1. Set \( u = x^2 + 1 \)
   compute \( du = 2x \, dx \)

2. Decomposition:
   \[
   \frac{6x}{(x^2 + 1)^3} \, dx = \frac{3 \cdot 2x}{u^3} \, dx = \frac{3}{u^3} \, du.
   \]

3. New integral: \( \int \frac{3}{u^3} \, du \)

4. Antiderivative using \( u \): \( = 3 \cdot \frac{u^{-2}}{-2} + C \)

5. Antiderivative using \( x \): \( = 3 \cdot \frac{(x^2 + 1)^{-2}}{-2} + C \)

**Example** \( \int x^2 e^{x^3 + 1} \, dx \).

1. Set \( u = x^3 + 1 \)
   compute \( du = 3x^2 \, dx \)

2. Decomposition:
   \[
x^2 e^{x^3 + 1} \, dx = 1 \cdot e^u \, du.
   \]

3. New integral: \( \int \frac{1}{3} e^u \, du \)

4. Antiderivative using \( u \): \( = \frac{1}{3} e^u + C \)

5. Antiderivative using \( x \): \( = \frac{1}{3} e^{x^3 + 1} + C \)

**Example** \( \int \frac{1}{1 - 4x} \, dx \).

Solution: Take \( u = 1 - 4x \) and compute the differential: \( du = -4 \, dx \). Since the function \( \frac{1}{1 - 4x} \) can be written as \( \frac{1}{u} \), we have to think about how to substitute \( dx \) in terms of \( du \). From the relation \( du = -4 \, dx \), we can divide both sides by \(-4\) to obtain: \( -\frac{1}{4} du = dx \). Thus the original integral turns to

\[
\int \frac{1}{1 - 4x} \, dx = \int \frac{1}{u} \cdot \left(-\frac{1}{4} \, du\right) = -\frac{1}{4} \int \frac{1}{u} \, du = -\frac{1}{4} \ln u + C \quad \text{plug in } u = 1 - 4x = -\frac{1}{4} \ln(1 - 4x) + C
\]
Model 1 Population model

\( y(t) \): The population at time \( t \).

\( y(0) \): Initial population.

\( y'(t) \): Rate of change of the population. (Population change from time \( t \) to \( t+1 \), due to birth, death and movement.)

1 Isolated environment, sufficient food supply:

Example 1.1. A lake is stocked with 1000 fish at beginning. Let \( y(t) \) be the number of fish after \( t \) months. Suppose that due to the new-born fish, during every month, the fish population will increase by 10%. Write a differential equation and determine the fish population after 12 months.

Answer. \( y' = 0.1y, y(0) = 1000 \).

2 Open environment, sufficient food supply.

Example 1.2 The birth rate in a certain city is 3.5% per year and the death rate is 2% per year. Also there is a net movement of population out of the city at a steady rate of 3000 people per year. Let \( y(t) \) denote the city's population at time \( t \). Write a differential equation satisfied by \( y(t) \). If the population of city now is 300,000. Write down the initial condition.

Answer. \( y' = 0.015y - 3000, y(0) = 300000 \).

Model 2 Savings account

\( y(t) \): The account balance at time \( t \).

\( y(0) \): Initial deposit.

\( y'(t) \): A description of balance change from time \( t \) to time \( t+1 \). (Due to interest, further deposit and withdrawal.)

Example 2.1. A company wishes to set aside funds for future expansion and so arranges to make a 10,000-dollar initial deposit into a savings account. The savings account earns 5% interest annually. Set up the differential equation and find the account balance after 5 years.

Answer. \( y' = 0.05y, y(0) = 10000 \).

Example 2.2. A company wishes to set aside funds for future expansion and so arranges to make a 10,000-dollar initial deposit into a savings account. The savings account earns 5% interest annually. Additionally, The company plans to make a 2,000-dollar constant deposit every year. Set up the differential equation and find the account balance after 5 years.

Answer. \( y' = 0.05y + 2000, y(0) = 10000 \).

Example 2.3. You took a loan of $ 25000 to pay for your a new car. The annual interest rate on the loan is 5%. You plan to make payments $ 4800 per year. Let \( y(t) \) be the amount that you owe on the loan in year \( t \). Set up an initial value problem that is satisfies by \( y(t) \).

Answer. \( y' = 0.05y - 4800, y(0) = 25000 \).
Submission

Problems to hand in: Problem 1, 2, 3, 4
Due: Week 2 (02/01 - 02/05) in your recitation

Problems

1. Differentiate the following functions using correct rules.
   (1) \( f(x) = x^4 - 3x^3 + \frac{x}{2} - \frac{3}{2} \); (2) \( f(x) = 4\sqrt{x} - \frac{4}{x^3} \); (3) \( y = \frac{3}{\sqrt{x}} - \sqrt{2} \).

2. Differentiate the following functions using correct rules.
   (1) \( y = x^2 \ln x \) (product rule); (2) \( f(x) = (3x^2 - \frac{4}{\sqrt{x}})^4 \) (chain rule); (3) \( g(t) = \frac{e^t}{t^2} \) (quotient rule); (4) \( f(x) = \sqrt{4x + 1} \) (chain rule); (5) \( y = e^{4x^2 + 3} \) (chain rule); (6) \( f(x) = \ln(5\sqrt{x} + 2) \) (chain rule).

3. Differentiate the following functions.
   (1) \( f(x) = (2x + 1)^3 e^{3x} \); (2) \( y = \left( \frac{1}{x^2 + 3} \right)^2 \); (3) \( f(x) = \ln \left( \frac{1}{\sqrt{x^2 + 1}} \right) \).

4. Compute the following indefinite integrals:
   (1) \( \int (4x^3 - 5\sqrt{x} + 6) \, dx \); (2) \( \int \left( \frac{3}{x^2} - \frac{1}{\sqrt{x}} \right) \, dx \); (3) \( \int \left( \frac{2e^{2x}}{3} + \frac{4}{x} \right) \, dx \); (4) \( \int \left( \frac{3x}{2} - \frac{2}{3x} \right) \, dx \).

5. (Combination of exponents. Review this topic on your own.) Simplify the following expressions by combining exponents of the same base. (Here \( x \) is assumed to be in the domain.)
   (1) \( x^3 \cdot \sqrt{x} \); (2) \( x^3 \cdot \frac{1}{\sqrt{x}} \); (3) \( \sqrt{x} \cdot \sqrt{x^3} \cdot x^2 \); (4) \( e^{4x} \cdot e^2 \); (5) \( \frac{(e^{3x})^2}{\sqrt{e}} \).
1 \( f'(x) = 4x^3 - 9x^2 + \frac{1}{2}; \) (2) \( f'(x) = 2x^{-1/2} + 12x^{-4}; \) (3) \( y' = -\frac{3}{2}x^{-3/2}. \)

2 (1) \( y' = 2x \ln x + x; \) (2) \( f'(x) = 4(3x^2 - \frac{4}{\sqrt{x}})^3 \cdot (6x + 2x^{-3/2}); \) (3) \( g'(t) = \frac{e^t \cdot t^2 - e^t \cdot 2t}{t^4}; \)

(4) \( f'(x) = \frac{1}{2}(4x + 1)^{-1/2} \cdot 4; \) (5) \( y' = e^{4x^2 + 3} \cdot 8x; \) (6) \( f'(x) = \frac{\frac{3}{2}x^{-1/2}}{5\sqrt{x} + 2}. \)

3 (1) \( f'(x) = 3(2x + 1)^2 \cdot 2 \cdot e^{3x} + (2x + 1)^3 \cdot e^{3x} \cdot 3; \) (2) Write \( y = (x^2 + 3)^{-2}, \) then \( y' = (-2) \cdot (x^2 + 3)^{-3} \cdot (2x); \)

(3) Write \( \frac{1}{\sqrt{x^2 + 1}} = (x^2 + 1)^{-1/2}. \) Then \( f'(x) = -\frac{1}{2}(x^2 + 1)^{-3/2} \cdot 2x = -\frac{x}{x^2 + 1}. \) (An easier way is to simplify \( f(x) \) using the log properties into \( f(x) = -\frac{1}{2} \ln(x^2 + 1).) \)

4 (1) \( x^4 - \frac{10}{3}x^{3/2} + 6x + C; \) (2) \( -\frac{3}{x} - 2\sqrt{x} + C; \) (3) \( \frac{1}{3}e^{2x} + 4 \ln x + C; \) (4) \( \frac{3}{4}x^2 - \frac{2}{3} \ln x + C. \)

5 (1) \( x^{25}; \) (2) \( x^{2.5}; \) (3) \( x^3; \) (4) \( e^{4x^2}; \) (5) \( e^{6x^2-\frac{1}{2}}. \)
Submission


Due: Week 3 (02/08 - 02/12) in your recitation

Problems

1. Use method of substitution to evaluate the indefinite integral \( \int (3x^2 + 1)(x^3 + x)^4 \, dx \) in steps:
   (1) Set up the substitution: \( u = x^3 + x \). Then compute the derivative of \( u \) with respect to \( x \): \( du = \) _______________ \( dx \).
   (2) Decompose the integral into a function of \( u \) and differential of \( u \):
   \( (3x^2 + 1)(x^3 + x)^4 \, dx = \) _______________ \( du \).
   (3) After substitution, we obtain a new simpler integral in terms of \( u \): \( \int \) _______________ \( du \);
   (4) Integrate the above new integral for the antiderivative. The antiderivative should be a function of \( u \):
   \( \int \) _______________ \( du = \) _______________ + \( C \);
   (5) Finally plug in the substitution to write the antiderivative as a function of \( x \):
   _______________.

2. Use substitution to compute the following indefinite integrals:
   (1) \( \int \frac{x^4}{(x^3 + 1)^2} \, dx \), substitution \( u = x^3 + 1 \);
   (2) \( \int (2x - 1)^7 \, dx \), substitution \( u = 2x - 1 \);
   (3) \( \int xe^{-x^2} \, dx \), substitution \( u = -x^2 \);
   (4) \( \int \frac{3x}{\sqrt{1-4x^2}} \, dx \), substitution \( u = 1 - 4x^2 \);
   (5) \( \int \frac{1}{2x + 1} \, dx \);
   (6) \( \int \frac{1}{2 - 3x} \, dx \);

3. Use integration by parts to compute the following integrals:
   (1) \( \int (3x + 1)e^{4x} \, dx \);
   (2) \( \int (x^2 + 2x)e^{3x} \, dx \); (apply integration by parts twice)
   (3) \( \int x^3 \ln x \, dx \);
   (4) \( \int x(1 - 2x)^5 \, dx \); (take \( f(x) = x \) and \( g(x) = (1 - 2x)^5 \), and compute \( G(x) \) using substitution method);
   (5) \( \int (\ln x)^2 \, dx \); (apply integration by parts twice)
   (6) \( \int x\sqrt{2x + 1} \, dx \).

4. (1) Use integration by parts to derive the following reduction formulas:
   \( \int x^n e^{kx} \, dx = \frac{x^n e^{kx}}{k} - \frac{n}{k} \int x^{n-1}e^{kx} \, dx \), for \( k \neq 0, n \geq 1 \)
(It is called a reduction formula because it transforms the integral involving \( x^n \) to one involving \( x^{n-1} \) and hence the degree of \( x \) in the integral reduces by one. So we can repeatedly use this formula to reduce the \( x \)-degree to zero.)

(2) Use the above formula to compute \( \int x^4 e^{2x} \, dx \).

5 Compute the following integrals using appropriate methods:

1. \( \int \frac{e^{x^2 + 5}}{\sqrt{2x + 5}} \, dx \) (use substitution \( u = \sqrt{2x + 5} \));
2. \( \int \frac{\ln(\ln x)}{x} \, dx \) (use substitution \( u = \ln x \));
3. \( \int \frac{x}{\sqrt{3x + 1}} \, dx \) (use integration by parts);
4. \( \int \frac{\ln x}{x^5} \) (use integration by parts).

**Answer** to Homework 2

**Error** If you find any error, please tell me: yinsu@buffalo.edu. Bonus points will be rewarded.

1. (1) \( 3x^2 + 1 \); (2) \( u^4 \, du \); (3) \( \int u^4 \, du \); (4) \( \int u^4 \, du = \frac{u^5}{5} + C \); (5) \( \frac{(x^3 + x)^5}{5} + C \).

2. (1) \( -\frac{1}{5}(x^5 + 1)^{-1} + C \); (2) \( = \frac{1}{16}(2x - 1)^8 + C \);
   (3) \( -\frac{1}{2} e^{-x^2} + C \); (4) \( \frac{3}{4} \sqrt{1 - 4x^2} + C \);
   (5) \( \frac{1}{3}(2x + 1)^{3/2} + C \); (6) \( -\frac{1}{3} \ln(2 - 3x) + C \).

3. (1) \( \frac{1}{4}(3x + 1)e^{3x} - \frac{3}{16} e^{4x} + C \); (2) \( \frac{1}{3}(x^2 + 2x)e^{3x} - \frac{1}{9}(2x + 2)e^{3x} + \frac{2}{27} e^{3x} + C \);
   (3) \( \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C \); (4) \( -\frac{1}{12} x(1 - 2x)^6 - \frac{1}{168}(1 - 2x)^7 + C \);
   (5) \( x(\ln x)^2 - 2x \ln x + 2x + C \); (6) \( \frac{1}{3}x(2x + 1)^{3/2} - \frac{1}{15}(2x + 1)^{5/2} + C \).

4. (1) Omit; (2) \( e^{2x} \left( \frac{1}{2} x^4 - x^3 + \frac{3}{2} x^2 - \frac{3}{2} x + \frac{3}{4} \right) + C \).

5. (1) \( e^{x^2 + 5} + C \); (2) \( \ln x \cdot \ln(\ln x) - \ln x + C \); (3) \( \frac{2}{3} x \sqrt{3x + 1} - \frac{4}{27}(3x + 1)^{3/2} + C \); (4) \( \frac{1}{4} x^{-4} \ln x - \frac{x^{-4}}{16} + C \).
Submission

Problems to hand in: Problem 2, 3 (2)(4)(6), 4 (1)(3)
Due: Week 5 (02/22 - 02/26) in your recitation

Problems

1. (1) Find the antiderivative of function $f(x) = \frac{4}{x^3}$.
   (2) Evaluate definite integral \( \int_{1}^{2} \frac{4}{x^3} \, dx \);
   (3) Let $b$ be a constant parameter. Evaluate definite integral \( \int_{1}^{b} \frac{4}{x^3} \, dx \);
   (4) Evaluate improper integral \( \int_{1}^{\infty} \frac{4}{x^3} \, dx \). Is it convergent or divergent?
   (5) Let $a$ be a constant parameter. Evaluate definite integral \( \int_{a}^{-2} \frac{4}{x^3} \, dx \);
   (6) Evaluate improper integral \( \int_{-\infty}^{-2} \frac{4}{x^3} \, dx \). Is it convergent or divergent?

2. (1) Find the antiderivative of function $f(x) = \frac{1}{\sqrt{4-x}}$ (using substitution $u = 4 - x$);
   (2) Let $a$ be a constant parameter. Evaluate the definite integral \( \int_{a}^{3} \frac{1}{\sqrt{4-x}} \, dx \);
   (3) Evaluate the improper integral \( \int_{-\infty}^{3} \frac{1}{\sqrt{4-x}} \, dx \). Is it convergent or divergent?

3. Evaluate the following improper integrals and determine if each of them is convergent or divergent.
   (1) \( \int_{0}^{\infty} xe^{-x^2} \, dx \) (use substitution $u = -x^2$);
   (2) \( \int_{0}^{\infty} 6e^{1-3x} \, dx \) (use substitution $u = 1 - 3x$);
   (3) \( \int_{-\infty}^{\infty} 6e^{1-3x} \, dx \); (use substitution $u = 1 - 3x$);
   (4) \( \int_{0}^{\infty} \frac{x^2}{(x^3 + 1)^3} \, dx \) (use substitution $u = x^3 + 1$).
   (5) \( \int_{1}^{\infty} \frac{3x}{\sqrt{x^2 + 3}} \, dx \); (use substitution $u = x^2 + 3$);
   (6) \( \int_{0}^{\infty} \frac{5x^2}{x^3 + 4} \, dx \) (use substitution $u = x^2 + 4$).
Use the midpoint rule to approximate the following definite integral by $n$ subintervals.

(1) $\int_{1}^{3} (x^2 + 1) \, dx$, for $n = 4$;

(2) $\int_{-2}^{2} (x^3 + 3x^2) \, dx$, for $n = 5$;

(3) $\int_{2}^{6} \sqrt{x^3 - 1}$ for $n = 5$. Round your answer to three decimal places.

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**Answer** to Homework 4

**Error** If you find any error, please tell me: yinsu@buffalo.edu. Bonus points will be rewarded.

1. (1) $-2x^{-2} + C$; (2) $\frac{3}{2}$; (3) $\frac{-2}{b^2} - \frac{-2}{1}$; (4) 2. Convergent; (5) $\left(\frac{-2}{(2)^2}\right) - \left(\frac{-2}{a^2}\right)$; (6) $-\frac{1}{2}$. Convergent.

2. (1) $-2\sqrt{4-x} + C$; (2) $(-2) - (-2\sqrt{4-a})$; (3) Divergent.

3. (1) $\frac{1}{2}$, convergent; (2) $2e$, convergent; (3) Divergent; (4) $\frac{1}{6}$, convergent; (5) divergent; (6) divergent.

4. (1) 10.625; (2) 15.360; (3) 32.670
Submission

Problems to hand in: Problem 2, 3, 4, 5, 7
Due: Week 6 (02/29 - 03/04) in your recitation

Problems

1. Use the midpoint rule to approximate the following definite integrals by \( n \) subintervals.
   
   (1) \( \int_{1}^{3} (x^2 + 1) \, dx \), for \( n = 4 \);  
   (2) \( \int_{-2}^{2} (x^3 + 3x^2) \, dx \), for \( n = 5 \);  
   (3) \( \int_{2}^{6} \sqrt{x^3 - 1} \) for \( n = 5 \). Round your answer to three decimal places.

2. Suppose we are given a differential equation \( y' = ty^2 \).
   
   (1) Check if \( y = e^{t^2} \) is a solution to this equation.
   
   (2) Check that \( y = \frac{2}{t^2 + C} \) is a solution to this equation for any constant \( C \). (Hint: write \( y = 2 \cdot (t^2 + C)^{-1} \) and apply the chain rule to find \( y' \).)
   
   (3) If we assume initial condition \( y(0) = 2 \), find a \( C \)-value in the solutions in part (2) and obtain a unique solution.
   
   (4) If we assume initial condition \( y(0) = -4 \), find the unique solution using the solutions in part (2).

3. Suppose we are given a differential equation \( y'' - 2y = 2ty' + 8t - 2 \).
   
   (1) Check if \( y = e^{t^2} \) is a solution to this equation.
   
   (2) Check that \( y = Ce^{t^2} - 2t + 1 \) is a solution to this equation for any constant \( C \). (Hint: compute the second derivative \( y''(t) \) using product rule.)
   
   (3) If we assume initial condition \( y(0) = 3 \), find the unique solution using the solutions in part (2).
   
   (4) If we assume initial condition \( y(0) = -2 \), find the unique solution using the solutions in part (2).

4. A lake is stocked with fish. Let \( y(t) \) be the number of fish after \( t \) months. Suppose the birth rate for the fish is 10% and the death rate is constantly 4%.
   
   (1) Set up a differential equation of \( y(t) \) representing the fish population.
   
   (2) Check that \( y = Ce^{0.06t} \) for any constant \( C \) is a solution to the differential equation obtained above.
   
   (3) Suppose at the beginning the fish population in the lake is 2000. Write this condition as an initial condition.
   
   (4) Use the solutions in part (2) to determine the value of \( C \). Then obtain a unique solution to this initial value problem.
   
   (5) What is the fish population in this lake after 12 months? Round your answer to an integer.
A population of insects in a region will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will increase with birth rate 50% per month. On any given day there is a net migration into the area of 30 insects and 15 are eaten by the local bird population and 5 die of natural causes. Let \( y(t) \) be the number of insects after \( t \) months and assume there are initially 100 insects in the area.

1. Write a differential equation with initial condition to represent the population \( y(t) \).
2. Check that \( y = 80e^{0.5t} - 20 \) is a solution to this initial value problem.
3. Using the solution in part (2), compute the number of insects after 6 months. Round your answer to an integer.

Suppose a family is depositing money into a bank account continuously. They begin their first year with $20,000 in the account and they make extra deposit at the rate of about $1,000 per year afterwards. Assume the account earns interest of 4% annually (compounded continuously) and the don't make any withdrawals. Let \( y(t) \) be the account balance after \( t \) years.

1. Write a differential equation with initial condition to represent the population \( y(t) \).
2. Check that \( y = 45000e^{0.04t} - 25000 \) is a solution to this initial value problem.
3. Using the solution in part (2), compute the account balance after 5 years. Round your answer to an integer.

Set up differential equations with initial condition for each of the following problems.

1. Suppose that you took out college loans totalling $60,000 with interest of 7.5%. You have a payment plan which continuously deducts money from your bank account at a rate which comes out to $15,000 per year. Let \( y(t) \) be the balance of the loan after \( t \) years. Write a differential equation with initial condition satisfied by \( y(t) \).
2. The depressed city of Rottentown has a current population of 300 thousand people, with a growth constant of 0.02. Over the following years, as news spreads about better opportunities elsewhere, there is to be an emigration rate of \( 2 + e^{0.03t} \) thousand people leaving the city per year, where \( t \) is the number of years from today. Let \( y(t) \) be the population (in thousands) after \( t \) years. Find an equation with initial condition which is satisfied by \( y(t) \).

Set up differential equations with initial condition for each of the following problems.

1. (Newton’s law of cooling.) A bath of water is kept at a constant temperature 10°C. A hot steel rod with initial temperature 110°C is plunged into the water bath. According to Newton’s law of cooling, the rate of change of the rod temperature after \( t \) seconds is 0.2 proportional to the difference between the bath temperature and the rod temperature. Set up a differential equation with initial condition satisfied by \( y(t) \), the temperature of the steel rod at time \( t \).
2. (Drug absorption.) A drug is introduced intravenously at a constant rate of 0.5mg/hour starting from some time spot called \( t = 0 \). On a continuous basis 2% of the drug is removed from the blood and absorbed by the body every hour. Write a differential equation with initial condition satisfied by \( y(t) \), the amount of drug in the patient.
Answer to Homework 4

Error If you find any error, please tell me: yinsu@buffalo.edu. Bonus points will be rewarded.

1 (1) 10.625; (2) 15.360; (3) 32.670

2 (1) No. (3) $C = 1$ and $y = \frac{2}{t^2 + 1}$. (4) $\frac{2}{t^2 - \frac{1}{2}}$.

3 (1) No. (2) Hint: $y'(t) = Ce^{t^2} \cdot 2t - 2$ and $y''(t) = Ce^{t^2} (4t^2 + 2)$.
   (3) $C = 2$ and $y = 2e^{t^2} - 2t + 1$; (4) $C = -3$ and $y = -3e^{t^2} - 2t + 1$.

4 (1) $y' = 0.06y$; (2) Answer omitted; (3) $y(0) = 2000$; (4) $C = 2000$ and $y = 2000e^{0.06t}$; (5) $2000e^{0.06 \cdot 12} \approx 4109$.

5 (1) $y' = 0.5y + 10$, $y(0) = 100$;
   (2) Hint: check this function satisfies the differential equation and the initial condition;
   (3) $80e^{0.5 \cdot 6} - 20 \approx 1587$.

6 (1) $y' = 0.04y + 1000$, $y(0) = 20000$;
   (2) Hint: check this function satisfies the differential equation and the initial condition;
   (3) $45000e^{0.04 \cdot 5} - 25000 \approx 29963$.

7 (1) $y' = 0.075y - 15000$, $y(0) = 60000$; (2) $y' = 0.02y - (2 + e^{0.03t})$, $y(0) = 300$.

8 (1) $y' = 0.2(110 - y)$, $y(0) = 10$; (2) $y' = 0.5 - 0.02y$, $y(0) = 0$. 

Problems

1 Differentiate the following functions:

(1) \[ y = (4x^3 + 2\sqrt{x} - 5)^9 \]

(2) \[ f(x) = (2x + 3)e^{4x} \]

Solution

(1) Identify \( u = (4x^3 + 2\sqrt{x} - 5) \) and then the whole function \( y = u^9 \). By chain rule we have

\[
y' = 9u^8 \cdot \frac{du}{dx} = 9(4x^3 + 2\sqrt{x} - 5)^8 \left( 4 \cdot 3x^2 + 2 \cdot \frac{1}{2}x^{-1/2} \right)
\]

Note that in the solution we write \( \sqrt{x} = x^{1/2} \).

(2) Identify two factors \( u(x) = 2x + 3 \) and \( v(x) = e^{4x} \). To use the product rule, we compute the derivatives first:

\[
u'(x) = 2, \quad v'(x) = e^{4x} \cdot 4 \text{ (by chain rule)}
\]

Then the product rule gives:

\[
f'(x) = u'v + uv' = 2e^{4x} + (2x + 3) \cdot e^{4x} \cdot 4.
\]

2 Compute the following indefinite integrals:

(1) \[ \int \left( \frac{3e^x}{5} - 6 \right) dx \]

(2) \[ \int \left( \frac{3}{x^2} - \frac{1}{3x} \right) dx \]

Solution

(1)

\[
\int \left( \frac{3e^x}{5} - 6 \right) dx = \int \frac{3e^x}{5} dx - \int 6dx = \frac{3}{5} \int e^x dx - \int 6dx = \frac{3}{5} e^x - 6x + C
\]

(2)

\[
\int \left( \frac{3}{x^2} - \frac{1}{3x} \right) dx = \int 3 \cdot x^{-2} dx - \int \frac{1}{3} \cdot \frac{1}{x} dx = 3 \int x^{-2} dx - \frac{1}{3} \int \frac{1}{x} dx = 3 \cdot x^{-1} - \frac{1}{3} \ln x + C
\]

Note that in the solution we write \( \frac{1}{x^2} = x^{-2} \) and hence \( \frac{3}{x^2} = 3 \cdot \frac{1}{x^2} = 3 \cdot x^{-2} \).
Problems

1. Differentiate the following functions:
   
   (1) \( y = (5x^6 + 3\sqrt{x} - 12)^7 \)
   
   (2) \( f(x) = \frac{3x - 4}{e^{4x}} \)

Solution

(1) Identify \( u = (5x^6 + 3\sqrt{x} - 12) \) and then the whole function \( y = u^7 \). By chain rule we have

\[
y' = 7u^6 \cdot \frac{du}{dx} = 7(5x^6 + 3\sqrt{x} - 12)^6 \cdot \left(5 \cdot 6x^5 + 3 \cdot \frac{1}{2} x^{-1/2}\right)
\]

Note that in the solution we write \( \sqrt{x} = x^{1/2} \).

(2) Identify two factors \( u(x) = 3x - 4 \) and \( v(x) = e^{4x} \). To use the quotient rule, we compute the derivatives first:

\[
u'(x) = 3, \quad v'(x) = e^{4x} \cdot 4 \text{ (by chain rule)}
\]

Then the product rule gives:

\[
f'(x) = \frac{u'v - uv'}{v^2} = \frac{3e^{4x} - (3x - 4) \cdot e^{4x} \cdot 4}{(e^{4x})^2} = \frac{3e^{4x} - 4(3x - 4)e^{4x}}{e^{8x}}.
\]

2. Compute the following indefinite integrals:

   (1) \( \int \left(\frac{3e^x}{5} + 3\sqrt{x} - 6\right) \, dx \)
   
   (2) \( \int \left(\frac{3}{\sqrt{x}} - \frac{2}{5x}\right) \, dx \)

Solution

(1)

\[
\int \left(\frac{3e^x}{5} + 3\sqrt{x} - 6\right) \, dx = \int \frac{3e^x}{5} \, dx + \int 3x^{1/2} \, dx - \int 6 \, dx = \frac{3}{5} \int e^x \, dx + 3 \int x^{1/2} \, dx - \int 6 \, dx = \frac{3}{5} e^{x} + \frac{2}{3} x^{3/2} - 6x + C
\]

(2)

\[
\int \left(\frac{3}{\sqrt{x}} - \frac{2}{5x}\right) \, dx = \int 3 \cdot x^{-1/2} \, dx - \int \frac{2}{5} \cdot \frac{1}{x} \, dx = 3 \int x^{-1/2} \, dx - \frac{2}{5} \int \frac{1}{x} \, dx = 3 \cdot 2x^{1/2} - \frac{2}{5} \ln x + C
\]

Note that in the solution we write \( \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2} \).

Write \( \frac{x}{\sqrt{3x + 1}} = x \cdot (3x + 1)^{-1/2} \). Call \( f(x) = x \) and \( g(x) = (3x + 1)^{-1/2} \). To compute \( G(x) \), we need to sub \( u = 3x + 1 \) with its derivative \( du = 3 \, dx \) and \( dx = \frac{1}{3} \, du \). Then

\[
G(x) = \int (3x + 1)^{-1/2} \, dx = \int u^{-1/2} \cdot \frac{1}{3} \, du = \frac{1}{3} \int u^{-1/2} \, du = \frac{1}{3} \cdot \frac{u^{1/2}}{1/2} = \frac{2}{3} (3x + 1)^{1/2}
\]
The goal now is to compute \( \int \frac{2}{3}(3x + 1)^{1/2} \, dx \). For this integral, again you need substitution \( u = 3x + 1 \) and the differential \( du = 3 \, dx \). This gives

\[
\int \frac{2}{3}(3x + 1)^{1/2} \, dx = \frac{2}{3} \int (3x + 1)^{1/2} \, dx \\
= \frac{2}{3} \int u^{1/2} \cdot \frac{1}{3} \, du \text{ (replace } dx \text{ by } \frac{1}{3} \, du) \\
= \frac{2}{3} \cdot \frac{1}{3} \int u^{1/2} \, du \text{ (Bring the constant } \frac{1}{3} \text{ out)} \\
= \frac{2}{9} \cdot \frac{u^{3/2}}{3/2} + C \\
= \frac{2}{9} \cdot \frac{2}{3} \cdot u^{3/2} + C \text{ (flip the } 2/3 \text{ to the numerator)} \\
= \frac{4}{27} \cdot (3x + 1)^{3/2} + C
\]
Exam 1 Information
Yin Su 2016.2.8

**Time and Location**: 02/15 10:00-10:50am Talbert 107

**Topics**:

1. Finding derivatives using all rules learned before (basic rules, combination rule, product and quotient rule, chain rule.)
2. Definition of indefinite integrals. Basic antiderivative rules.
4. Integration by parts. Combine integration by parts with substitution method in one integral.

**Total points**: 70 points, including a bonus problem of 10 points.

**What to bring**

- **Needed**: Pen/Pencil, Photo ID.
- **Allowed**: Non-graphing scientific calculator.
- **Not allowed**: Textbook, notes, other references.

You will get in exam: a formula sheet. (See next page. Also available on UBLearns. You don't need to print it out.)

**Practice**:

1. A practice exam is on UBLearns with answer. Full solution will be available on UBLearns Thursday 02/11.
2. Homework coverage: Homework 1 to 2.

**Remarks**:

1. If you need any special accommodation, please tell me ASAP.
2. You need an official excuse if you miss the exam and want to make it up. The makeup exam will be harder than the usual one and only has 60 points in total.
3. To avoid cheating, two versions of exams will be used.
• Basic differentiation and integration rules

<table>
<thead>
<tr>
<th>Differentiation</th>
<th>Integration</th>
</tr>
</thead>
<tbody>
<tr>
<td>(constant)′ = 0</td>
<td>∫ k dx = kx + C for constant k</td>
</tr>
<tr>
<td>(x^n)′ = nx^{n-1}</td>
<td>∫ x^n dx = \frac{x^{n+1}}{n+1} + C, (n ≠ -1)</td>
</tr>
<tr>
<td>(e^x)′ = e^x</td>
<td>∫ e^{kx} dx = \frac{1}{k}e^{kx} + C, (k ≠ 0)</td>
</tr>
<tr>
<td>(\ln x)′ = \frac{1}{x}</td>
<td>∫ \frac{1}{x} dx = \ln x + C</td>
</tr>
</tbody>
</table>

• Product rule and quotient rule for differentiation:

\[(fg)' = f' \cdot g + f \cdot g'\]
\[\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}\]

• Integration by parts:

\[\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)dx\]

• Fundamental theorem of calculus:

\[\int_a^b f(x)dx = F(b) - F(a)\]

where \(F(x)\) is an antiderivative of \(f(x)\).
1 Find the derivative of the following functions
(1) \( f(x) = \ln(5x^2 + 4x - 5) \) \hspace{1cm} (2) \( f(x) = e^{6x} \sqrt{1 - 2x} \).

2 Compute the following integrals \( \int \left( \frac{2e^{3x}}{3} + \frac{4}{\sqrt{x}} - \frac{3}{5x} \right) \, dx \).

3 Compute the following integral using the method of substitution: \( \int 3x^2e^{x+1} \, dx \).

4 Compute the following integral using integration by parts: \( \int x \sqrt{3x + 2} \, dx \).

5 Compute the area of the shaded region shown in the figure. The region is bounded above by the graph of the function \( f(x) = \frac{3x}{x^2 + 1} \) between \( x = 0 \) and \( x = 3 \).

6 Finish the following questions with short answer.
(1) True or false: Every continuous function has a unique antiderivative function.
(2) True or false: The derivative of \( \ln 5 \) is \( \frac{1}{5} \).
(3) For the integral \( \int x^2 \ln x \, dx \), to correctly use the formula of integration by parts: \( \int f(x)g(x) \, dx = f(x)G(x) - \int f'(x)G(x) \, dx \), which part should we choose as \( f(x) \)? Answer: \( x^2 \).
(4) Find a function whose second derivative is \( e^{3x} \). Answer: \( e^{3x} \).
(5) In the following figure, region A has area 3 units and region B has 4 units. Then the integral \( \int_0^5 f(x) \, dx \) equals ________.

7 (Bonus problem) Compute the following integral \( \int \frac{x^3}{e^{x^2+1}} \, dx \).
1. \( f'(x) = \frac{10x + 4}{5x^2 + 4x - 5} \)
2. \( f'(x) = 6e^{6x} \cdot \sqrt{1 - 2x} + e^{6x} \cdot \frac{1}{2}(1 - 2x)^{-1/2} \cdot (-2) \)
3. \( \frac{2}{9}e^{3x} + 8\sqrt{x} - \frac{3}{5} \ln x + C \)
4. \( \frac{1}{2}x^x + 1 + C \)
5. \( \frac{2}{9}(3x + 2)^{3/2} - \frac{4}{135}(3x + 2)^{5/2} + C \)
6. \( \int_0^3 \frac{3x}{x^2 + 1} \, dx = \frac{3}{2} \ln 10 \)
7. \( -\frac{1}{2}(x^2 + 1)e^{-(x^2 + 1)} + C \)
1. Find the derivative of the following functions
   (1) \( f(x) = \ln(5x^2 + 4x - 5) \)
   (2) \( f(x) = e^{6x} \sqrt{1-2x} \)

   **Solution**
   
   (1) Take \( u = 5x^2 + 4x - 2 \) as the inside function. Then whole function \( y = \ln u \). By chain rule, we have
   \[
   f'(x) = \frac{1}{u} \cdot u'(x) = \frac{10x + 4}{5x^2 + 4x - 2}
   \]
   
   (2) Note that \( y \) is product function. We identify the factors \( f(x) = e^{6x} \) and \( g(x) = \sqrt{1-2x} \) and we want to apply the product rule. We compute the derivatives of \( f \) and \( g \) first, using chain rule:
   
   \[
   f'(x) = e^{6x} \cdot 6, \quad g'(x) = \frac{1}{2}(1-2x)^{-1/2} \cdot (-2)
   \]
   Notice that we rewrite \( g(x) = (1-2x)^{1/2} \) and apply the chain rule to obtain the derivative \( g'(x) \).
   
   Then by product rule
   \[
   y' = f'g + fg' = 6e^{6x} \cdot \sqrt{1-2x} + e^{6x} \cdot \frac{1}{2}(1-2x)^{-1/2} \cdot (-2)
   \]

2. Compute the following integrals \( \int \left( \frac{2e^{3x}}{3} + \frac{4}{\sqrt{x}} - \frac{3}{5x} \right) dx \).

   **Solution**
   
   Notice that the function to be integrated is a combination of three components, which are connected by addition and subtraction. Thus the integral can split into three.
   
   \[
   \int \left( \frac{2e^{3x}}{3} + \frac{4}{\sqrt{x}} - \frac{3}{5x} \right) dx = \int \frac{2e^{3x}}{3} dx + \int \frac{4}{\sqrt{x}} dx - \int \frac{3}{5x} dx
   \]
   
   \[
   = \frac{2}{3} \int e^{3x} dx + 4 \int x^{-1/2} dx - \frac{3}{5} \int \frac{1}{x} dx \quad \text{(bring constant multiples out of the integrals)}
   \]
   
   \[
   = \frac{2}{3} \cdot \frac{e^{3x}}{3} + 4 \cdot \frac{x^{1/2}}{1/2} - \frac{3}{5} \ln x + C
   \]

3. Compute the following integral using the method of substitution: \( \int 3x^5 e^{x^6+1} dx \).

   **Solution**
   
   Take \( u = x^6 + 1 \) and compute its derivative: \( du = 6x^5 dx \) by power rule. To complete the substitution, we notice that the function in the integral has \( x^5 dx \) in the numerator, which also shows up in the differential relation. Thus we divide 6 in the differential relation obtain:
   
   \[
   du = 6x^5 dx \quad \Rightarrow \quad \frac{1}{6} du = x^5 dx
   \]
Then we apply the substitution:

\[
\int 3 \cdot x^5 e^{x^6 + 1} \, dx = \int 3 \cdot e^u \cdot \frac{1}{2} \, du
\]

\[
= \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^6 + 1} + C
\]

4 Compute the following integral using integration by parts: \( \int x \sqrt{3x + 2} \, dx \).

**Solution** Identify \( f(x) = x \) and \( g(x) = \sqrt{3x + 2} = (3x + 2)^{1/2} \). Then we have \( f'(x) = 1 \). To get an antiderivative \( G(x) \), we need to apply substitution \( u = 3x + 1 \) and \( du = 3 \, dx \):

\[
G(x) = \int (3x + 2)^{1/2} \, dx = \int u^{1/2} \cdot \frac{1}{3} \, du = \frac{1}{3} \frac{u^{3/2}}{3/2} = \frac{2}{9} (3x + 2)^{3/2}
\]

Then by integration by parts, we have

\[
\int x \sqrt{3x + 2} \, dx = x \cdot \frac{2}{9} (3x + 2)^{3/2} - \frac{1}{3} \int \frac{1}{9} (3x + 2)^{3/2} \, dx
\]

\[
= x \cdot \frac{2}{9} (3x + 2)^{3/2} - \frac{1}{9} \left( \frac{1}{3} \frac{(3x + 2)^{5/2}}{5/2} \right) + C \text{ (evaluate the remaining integral using substitution)}
\]

\[
= x \cdot \frac{2}{9} (3x + 2)^{3/2} - \frac{4}{135} (3x + 2)^{5/2} + C
\]

where the last integral using \( u = 3x + 2 \) and \( du = 3 \, dx \) goes as follows:

\[
\int (3x + 2)^{3/2} \, dx = \int u^{3/2} \cdot \frac{1}{3} \, du = \frac{1}{3} \frac{u^{5/2}}{5/2} + C = \frac{1}{3} \frac{(3x + 2)^{5/2}}{5/2} + C
\]

5 Compute the area of the shaded region shown in the figure. The region is bounded above by the graph of the function \( f(x) = \frac{3x}{x^2 + 1} \, dx \) between \( x = 0 \) and \( x = 3 \).

![Graph of f(x)](image)

**Solution** By the geometric definition of definite integrals, we can realize the area of the shaded region as the integral

\[
\int_0^3 \frac{3x}{x^2 + 1} \, dx
\]

This is a definite integral. The first step on a definite integral is to compute the antiderivative. Thus we use substitution \( u = x^2 + 1 \) with its differential \( du = 2x \, dx \). The differential implies \( \frac{1}{2} du = x \, dx \). Using this we can transform the integral:

\[
\int \frac{3x}{x^2 + 1} \, dx = \int 3 \cdot \frac{1}{x^2 + 1} \cdot x \, dx = \int 3 \cdot \frac{1}{u} \cdot \frac{1}{2} \, du = \frac{3}{2} \int \frac{1}{u} \, du = \frac{3}{2} \ln u + C = \frac{3}{2} \ln(x^2 + 1) + C
\]
So we can take an antiderivative function to be $F(x) = \frac{3}{2} \ln(x^2 + 1)$.

Then the second step is to evaluate the antiderivative function $F(x)$ over the bounds:

$$\int_0^3 \frac{3x}{x^2 + 1} dx = \left. F(x) \right|_{x=0}^{x=3} = \frac{3}{2} \ln(x^2 + 1) \bigg|_{x=0}^{x=3}$$

$$= \left( \frac{3}{2} \ln(3^2 + 1) \right) - \left( \frac{3}{2} \ln(0^2 + 1) \right) = \frac{3}{2} \ln 10 - \frac{3}{2} \ln 1 = \frac{3}{2} \ln 10$$

Therefore the area of the shaded region is $\frac{3}{2} \ln 10$.

6 Finish the following questions with short answer.

(1) True or false: Every continuous function has a unique antiderivative function.

(2) True or false: The derivative of $\ln 5$ is $\frac{1}{5}$.

(3) For the integral $\int x^2 \ln x dx$, to correctly use the formula of integration by parts: $\int f(x)g(x)dx = f(x)G(x) - \int f(x)G(x)dx$, which part should we choose as $f(x)$? Answer: ____________

(4) Find a function whose second derivative is $e^{3x}$. Answer: ____________

(5) In the following figure, region A has area 3 units and region B has 4 units. Then the integral $\int_0^5 f(x)dx$ equals ____________.

---

**Solution**

(1) False. If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ for any constant $C$ is also an antiderivative. Thus actually there are infinitely many antiderivative functions to a given function $f(x)$.

(2) False. $\ln 5$ is a constant because 5 is a fixed constant (not a variable). Its derivative is actually zero. ($\ln x$ is a function of the variable $x$ with derivative $\frac{1}{x}$.)

(3) We should take $f(x) = \ln x$.

(4) If we need a function with second derivative $e^{3x}$, we just integrate $e^{3x}$ twice. Integrate once we have $\frac{1}{3} e^{3x} + C$ for some constant $C$. Integrate this function again. we have $\frac{1}{9} e^{3x} + Cx + D$ for arbitrary constants $C$ and $D$. So any function of this form will be a right answer. For example, $\frac{1}{9} e^{3x}$, or $\frac{1}{9} e^{3x} + 2x + 3$.

(5) By the geometric definition of definite integrals, the value of a definite integral is the net area of regions enclosed by the curve between bounds. In the figure, between $x = 0$ and $x = 5$, the function encloses two regions $A$ and $B$ with area 3 and 4. But region $A$ is below $x-$axis. Thus the value of the integral is $\int_0^5 f(x)dx = -\text{Area}(A) + \text{Area}(B) = -3 + 4 = 1$.
(Bonus problem) Compute the following integral \( \int \frac{x^3}{e^{x^2+1}} \, dx \).

\textbf{Solution} Firstly take a substitution \( u = x^2 + 1 \) with derivative \( du = 2x \, dx \). To transform the integral we write \( x^3 \) as \( x^2 \cdot x \) and \( x^2 = u - 1 \). Then

\[
\int \frac{x^3}{e^{x^2+1}} \, dx = \int x^2 \cdot e^{-(x^2+1)} \cdot x \, dx = \int (u-1)e^{-u} \cdot \frac{1}{2} \, du = \frac{1}{2} \int (u-1)e^{-u} \, dx
\]

The resulting integral has a product structure. So we take \( f(u) = u - 1 \) and \( g(u) = e^{-u} \). Then \( f'(u) = 1 \) and \( G(u) = -e^u \). By integration by parts the above resulting integral equals

\[
\frac{1}{2} \int (u-1)e^{u} \, dx = \frac{1}{2} \left( (u-1) \cdot (-e^{u}) - \int 1 \cdot (-e^{u}) \, du \right) + C
\]

\[
= \frac{1}{2} \left( -(u-1)e^{u} + \int e^{u} \, dx \right) + C
\]

\[
= \frac{1}{2} \left( -(u-1)e^{u} - e^{u} \right) + C
\]

\[
= \frac{1}{2} \left( -x^2 e^{-(x^2+1)} - e^{-(x^2+1)} \right) + C
\]

\[
= \frac{1}{2} \left( x^2 + 1 \right) e^{-(x^2+1)} + C
\]
1. Find the derivative of the following functions

(1) \( y = \ln(x^2 + \frac{2}{x}) \)

Solution

(1) Take \( u = x^2 + \frac{2}{x} = x^2 + 2x^{-1} \) as the inside function. Then whole function \( y = \ln u \). By chain rule, we have

\[
 f'(x) = \frac{1}{u} \cdot u'(x) = \frac{2x + 2 \cdot (-1)x^{-2}}{x^2 + 2x^{-1}}
\]

(2) Note that \( y \) has a chain structure. We take \( u = \frac{x^2 + 1}{e^{6x}} \). Then \( y = u^3 \). Then we apply the chain rule to compute the derivative function.

\[
 y' = 3u^2 \cdot u'(x) = 3 \left( \frac{x^2 + 1}{e^{6x}} \right)^2 \cdot \frac{2x \cdot e^{6x} - (x^2 + 1) \cdot (e^{6x} \cdot 6)}{(e^{6x})^2}
\]

In the above computation, to find \( u'(x) \), we applied the quotient rule.

2. Compute the following integral \( \int \left( \frac{1}{3} - 3\sqrt{x} + \frac{2}{3x} \right) dx \).

Solution

Notice that the function to be integrated is a combination of three components, which are connected by addition and subtraction. Thus the integral can split into three.

\[
 \int \left( \frac{1}{3} - 3\sqrt{x} + \frac{2}{3x} \right) dx = \int \frac{1}{3} dx - 3 \int \sqrt{x} dx + \int \frac{2}{3x} dx
\]

\[
 = \frac{1}{3} x - 3 \int x^{1/2} dx + \frac{2}{3} \int \frac{1}{x} dx \quad \text{(bring constant multiples out of the integrals)}
\]

\[
 = \frac{1}{3} x - 3 \cdot \frac{x^{3/2}}{3/2} + \frac{2}{3} \ln x + C
\]

3. Compute the following integral using the method of substitution: \( \int \frac{5x^2}{(x^3 + 4)^3} dx \).

Solution

Take \( u = x^3 + 4 \) and compute its derivative: \( du = 3x^2 dx \) by power rule. To complete the substitution, we notice that the function in the integral has \( x^2 dx \) in the numerator, which also shows up in the differential relation. Thus we divide 3 in the differential relation obtain:

\[
 du = 3x^2 dx \quad \Rightarrow \quad \frac{1}{3} du = x^2 dx
\]

Then we apply the substitution:

\[
 \int \frac{5x^2}{(x^3 + 4)^3} dx = \int 5 \cdot \frac{1}{u^3} \cdot \frac{1}{3} du
\]

\[
 = \frac{5}{3} \int u^{-3} du = \frac{5}{3} \cdot \frac{u^{-2}}{-2} + C = \frac{5}{3} \cdot \frac{(x^3 + 4)^{-2}}{-2} + C
\]
4. Compute the following integral using integration by parts: \( \int (x^2 + 2x)e^{2x} \, dx \). (Hint: You need to apply integration by parts twice.)

**Solution** Identify \( f(x) = x^2 + 2x \) and \( g(x) = e^{2x} \). Then we have \( f'(x) = 2x + 2 \) and \( G(x) = \frac{1}{2}e^{2x} \). Then by integration by parts, we have

\[
\int (x^2 + 2x)e^{2x} \, dx = (x^2 + 2x) \cdot \frac{1}{2}e^{2x} - \int (2x + 2) \cdot \frac{1}{2}e^{2x} \, dx \text{ (formula of integration by parts)}
\]

\[
= \frac{1}{2}(x^2 + 2x)e^{2x} - \frac{1}{2} \int (2x + 2)e^{2x} \, dx \text{ (pull out constant multiples)}
\]

\[
\text{Integral (I)}
\]

On the right hand side, we still see an integral, which involves a product function. Thus we want to apply integration by parts again. For this purpose, we identify \( f(x) = (2x + 2) \) and \( g(x) = e^{2x} \). Then \( f'(x) = 2 \) and \( G(x) = \frac{1}{2}e^{2x} \), and by formula, the integral (I) equals:

\[
\text{(I)} = \int (2x + 2)e^{2x} \, dx = (2x + 2) \cdot \frac{1}{2}e^{2x} - \int 2 \cdot \frac{1}{2}e^{2x} \, dx
\]

\[
= (2x + 2) \cdot \frac{1}{2}e^{2x} - \int e^{2x} \, dx
\]

\[
= (2x + 2) \cdot \frac{1}{2}e^{2x} - \frac{1}{2}e^{2x} + C
\]

Therefore, substitute the above answer of integral (I), we have the answer of the initial integral:

\[
\int (x^2 + 2x)e^{2x} \, dx = \frac{1}{2}(x^2 + 2x)e^{2x} - \frac{1}{2} \left( (2x + 2) \cdot \frac{1}{2}e^{2x} - \frac{1}{2}e^{2x} + C \right)
\]

\[
= \frac{1}{2}(x^2 + 2x)e^{2x} - \frac{1}{4}(2x + 2)e^{2x} + \frac{1}{4}e^{2x} + C
\]

5. Compute the area of the shaded region shown in the figure. The region is bounded above by the graph of the function \( f(x) = \frac{3}{\sqrt{2x + 4}} \) between \( x = 0 \) and \( x = 6 \).

**Solution** By the geometric definition of definite integrals, we can realize the area of the shaded region as the integral

\[
\int_0^6 \frac{3}{\sqrt{2x + 4}} \, dx
\]

This is a definite integral. The first step on a definite integral is to compute the antiderivative. Thus we use substitution \( u = 2x + 4 \) with its differential \( du = 2 \, dx \).

The differential implies \( \frac{1}{2} \, du = dx \). Using this we can transform the integral:

\[
\int \frac{3}{\sqrt{2x + 4}} \, dx = \int \frac{3}{\sqrt{u}} \cdot \frac{1}{2} \, du = \frac{3}{2} \int u^{-1/2} \, du
\]

\[
= \frac{3}{2} \frac{u^{1/2}}{1/2} = 3u^{1/2} + C = 3\sqrt{2x + 4} + C
\]

So we can take an antiderivative function to be \( F(x) = 3\sqrt{2x + 4} \).
Then the second step is to evaluate the antiderivative function \( F(x) \) over the bounds:

\[
\int_{0}^{6} \frac{3}{\sqrt{2x+4}} \, dx = F(x)\bigg|_{x=0}^{x=6} = 3\sqrt{2x+4}\bigg|_{x=0}^{x=6} = (3\sqrt{2 \cdot 6 + 4}) - (3\sqrt{2 \cdot 0 + 4}) = 12 - 6 = 6
\]

Therefore the area of the shaded region is 6.

6 Finish the following questions with short answer.

(1) The derivative of \( \frac{1}{x} \) is \( -\frac{1}{x^2} \).

(2) Which of the following is correct? Only one choice is correct.

(A) Any two antiderivative functions of a fixed function \( f(x) \) can only differ by a constant.

(B) The antiderivative of \( e^{-x} \) is \( e^{-x} + C \).

(C) The derivative of \( \frac{x^2}{x^3 + x} \) is \( \frac{2x}{3x^2 + 1} \).

(D) The antiderivative of \( \frac{1}{x^2} \) is \( \ln(x^2) + C \).

(3) Which of the following integrals can be computed using the method of substitution?

(A) \( \int x^2 \ln x \, dx \); (B) \( \int xe^{2x} \, dx \); (C) \( \int x \sqrt{x + 1} \, dx \); (D) \( \int x \sqrt{x^2 + 1} \, dx \).

(4) Find a function whose second derivative is \( \frac{1}{x^2} \). Your answer: \( -\ln x \) or \( -\ln x + 2x + 3 \).

(5) In the figure on the right, the function \( f(x) \) encloses several regions with \( x \)-axis. The area for the indicated regions is listed as follows:

<table>
<thead>
<tr>
<th>Region</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>3</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
</tr>
</tbody>
</table>

Then the definite integral \( \int_{2}^{8} f(x) \, dx \) equals \( -2 \).

**Solution**

(1) Write \( \frac{1}{x} \) as \( x^{-1} \). Then by power rule, its derivative is \( -x^{-2} = -\frac{1}{x^2} \).

(2) (A)

(3) (D) All other three choices have to use integration by parts.

(4) If we need a function with second derivative \( \frac{1}{x^2} \), we just integrate \( \frac{1}{x^2} \) twice. Write \( \frac{1}{x^2} = x^{-2} \). Integrate once. We have \( -\frac{1}{x} + C \) for some constant \( C \). Integrate this function again. We have \( -\ln x + Cx + D \) for arbitrary constants \( C \) and \( D \). So any function of this form will be a right answer. For example, \( -\ln x \) or \( -\ln x + 2x + 3 \).

(5) By the geometric definition of definite integrals, the value of a definite integral is the net area of regions enclosed by the curve between bounds. In the figure, between \( x = 2 \) and \( x = 8 \), the function encloses three regions \( B, C \) and \( D \). Notice regions \( B \) and \( D \) are below \( x \)-axis. Thus the value of the integral is

\[
\int_{2}^{8} f(x) \, dx = -\text{Area}(B) + \text{Area}(C) - \text{Area}(D) = -5 + 4 - 3 = -2
\]
Bonus Problem: Compute the following integral \( \int \sqrt{1 + \sqrt{x}} \, dx \).

Solution: Take substitution \( u = \sqrt{x} \). Instead of directly differentiating it, we rewrite the substitution as \( x = u^2 \) and differentiate this relation by \( u \) to obtain:

\[
x = u^2 \quad \Rightarrow \quad dx = 2u \, du
\]

Then the original integral can be transformed into

\[
\int \sqrt{1 + \sqrt{x}} \, dx = \int \sqrt{1 + u} \cdot 2u \, du
\]

Now we are going to use integration by parts. Take \( f(u) = 2u \) and \( g(u) = \sqrt{1 + u} = (1 + u)^{1/2} \). Then we have

\[
f'(u) = 2 \quad \quad G(u) = \int (1 + u)^{1/2} = \frac{2}{3}(1 + u)^{3/2}
\]

Thus the resulting above integral now equals:

\[
\int \sqrt{1 + u} \cdot 2u \, du = 2u \cdot \frac{2}{3}(1 + u)^{3/2} - \int \frac{2}{3}(1 + u)^{3/2} \, du
\]

\[
= \frac{4}{3}u(1 + u)^{3/2} - \frac{4}{3} \int (1 + u)^{3/2} \, du
\]

\[
= \frac{4}{3}u(1 + u)^{3/2} - \frac{4}{3} \cdot \frac{2}{5}(1 + u)^{5/2} + C
\]

\[
= \frac{4}{3} \sqrt{x}(1 + \sqrt{x})^{3/2} - \frac{8}{15}(1 + \sqrt{x})^{5/2} + C
\]
1 Find the derivative of the following functions

(1) \( y = \left( \frac{e^{6x}}{e^{x^2} + 2x} \right)^3 \) 

\( \textbf{Solution} \)

(1) Note that \( y \) has a chain structure. We take \( u = \frac{e^{6x}}{x^3 + 2x} \). Then \( y = u^3 \). Then we apply the chain rule to compute the derivative function.

\[
y' = 3u^2 \cdot u'(x) = 3 \left( \frac{e^{6x}}{x^3 + 2x} \right)^2 \cdot \left( \frac{e^{6x} \cdot 6 \cdot (x^3 + 2x) - e^{6x} \cdot (3x^2 + 2)}{(x^3 + 2x)^2} \right)
\]

In the above computation, to find \( u'(x) \), we applied the quotient rule.

(2) \( y = \ln(x^3 + 5 \sqrt{x}) \)

\( \textbf{Solution} \)

(2) Take \( u = x^3 + 5 \sqrt{x} = x^3 + 5x^{1/2} \) as the inside function. Then whole function \( y = \ln u \). By chain rule, we have

\[
f'(x) = \frac{1}{u} \cdot u'(x) = \frac{3x^2 + 5 \cdot \frac{1}{2} x^{-1/2}}{x^3 + 5x^{1/2}}
\]

2 Compute the following integral \( \int \left( \frac{1}{3} - 3 \sqrt{x} + \frac{2}{3x} \right) dx \).

\( \textbf{Solution} \)

Notice that the function to be integrated is a combination of three components, which are connected by addition and subtraction. Thus the integral can split into three.

\[
\int \left( \frac{1}{3} - 3 \sqrt{x} + \frac{2}{3x} \right) dx = \int \frac{1}{3} dx - \int 3 \sqrt{x} dx + \int \frac{2}{3x} dx
\]

\[
= \frac{1}{3} x - 3 \int x^{1/2} dx + \frac{2}{3} \int \frac{1}{x} dx \quad \text{(bring constant multiples out of the integrals)}
\]

\[
= \frac{1}{3} x - 3 \cdot \frac{x^{3/2}}{3/2} + \frac{2}{3} \ln x + C
\]

3 Compute the following integral using the method of substitution: \( \int \frac{2x^2}{(x^3 - 3)^5} dx \).

\( \textbf{Solution} \) Take \( u = x^3 - 3 \) and compute its derivative: \( du = 3x^2 dx \) by power rule. To complete the substitution, we notice that the function in the integral has \( x^2 dx \) in the numerator, which also shows up in the differential relation. Thus we divide 3 in the differential relation obtain:

\[
du = 3x^2 dx \quad \Rightarrow \quad \frac{1}{3} du = x^2 dx
\]

Then we apply the substitution:

\[
\int \frac{2x^2}{(x^3 - 3)^5} dx = \int 2 \cdot \frac{1}{u^5} \cdot \frac{1}{3} du = \frac{2}{3} \int u^{-5} du = \frac{2}{3} \cdot \frac{u^{-4}}{-4} + C = \frac{2}{3} \cdot \frac{(x^3 - 3)^{-4}}{-4} + C = -\frac{1}{6} (x^3 - 3)^{-4} + C
\]

4 Compute the following integral using integration by parts: \( \int (x^2 + 2x)e^{2x} dx \). (Hint: You need to apply integration by parts twice.)
**Solution** Identify \( f(x) = x^2 + 2x \) and \( g(x) = e^{2x} \). Then we have \( f'(x) = 2x + 2 \) and \( G(x) = \frac{1}{2} e^{2x} \). Then by integration by parts, we have

\[
\int (x^2 + 2x)e^{2x}\,dx = (x^2 + 2x)\cdot \frac{1}{2} e^{2x} - \int (2x + 2)\cdot \frac{1}{2} e^{2x}\,dx \quad \text{formula of integration by parts}
\]

\[
= \frac{1}{2} (x^2 + 2x)e^{2x} - \frac{1}{2} \int (2x + 2)e^{2x}\,dx \quad \text{pull out constant multiples}
\]

On the right hand side, we still see an integral, which involves a product function. Thus we want to apply integration by parts again. For this purpose, we identify \( f(x) = 2x + 2 \) and \( g(x) = e^{2x} \). Then \( f'(x) = 2 \) and \( G(x) = \frac{1}{2} e^{2x} \), and by formula, the integral (I) equals:

\[
(I) = \int (2x + 2)e^{2x}\,dx = (2x + 2)\cdot \frac{1}{2} e^{2x} - \int 2\cdot \frac{1}{2} e^{2x}\,dx
\]

\[
= (2x + 2)\cdot \frac{1}{2} e^{2x} - \int e^{2x}\,dx
\]

\[
= (2x + 2)\cdot \frac{1}{2} e^{2x} - \frac{1}{2} e^{2x} + C
\]

Therefore, substitute the above answer of integral (I), we have the answer of the initial integral:

\[
\int (x^2 + 2x)e^{2x}\,dx = \frac{1}{2} (x^2 + 2x)e^{2x} - \frac{1}{2} \int (2x + 2)e^{2x}\,dx
\]

\[
= \frac{1}{2} (x^2 + 2x)e^{2x} - \frac{1}{2} \left( (2x + 2)\cdot \frac{1}{2} e^{2x} - \frac{1}{2} e^{2x} + C \right)
\]

\[
= \frac{1}{2} (x^2 + 2x)e^{2x} - \frac{1}{4} (2x + 2)e^{2x} + \frac{1}{4} e^{2x} + C
\]

**5** Compute the area of the shaded region shown in the figure. The region is bounded above by the graph of the function \( f(x) = \frac{5}{\sqrt{3x + 4}} \) between \( x = 0 \) and \( x = 4 \).

**Solution** By the geometric definition of definite integrals, we can realize the area of the shaded region as the integral

\[
\int_0^4 \frac{5}{\sqrt{3x + 4}}\,dx
\]

This is a definite integral. The first step on a definite integral is to compute the antiderivative. Thus we use substitution \( u = 3x + 4 \) with its differential \( du = 3\,dx \).

The differential implies \( \frac{1}{3}du = dx \). Using this we can transform the integral:

\[
\int \frac{5}{\sqrt{3x + 4}}\,dx = \int \frac{5}{\sqrt{u}} \cdot \frac{1}{3}du = \frac{5}{3} \int u^{-1/2}du = \frac{5}{3} \cdot \frac{1}{2}u^{1/2} + C = \frac{5}{3} \sqrt{3x + 4} + C
\]

So we can take an antiderivative function to be \( F(x) = \frac{10}{3} \sqrt{3x + 4} \).

Then the second step is to evaluate the antiderivative function \( F(x) \) over the bounds:

\[
\left[ \frac{5}{\sqrt{3x + 4}} \right]_0^4 = \frac{10}{3} \sqrt{3 \cdot 4} + 4 - \frac{10}{3} \sqrt{3 \cdot 0 + 4} = \frac{10}{3} \cdot 4 - \frac{10}{3} \cdot 2 = \frac{20}{3}
\]
Therefore the area of the shaded region is $\frac{20}{3}$.

6. Finish the following questions with short answer.

(1) Which of the following integrals can be computed WITHOUT using integration by parts? Only one choice is correct.

(A) $\int x^2 \ln x \, dx$; (B) $\int x \sqrt{x^2 + 1} \, dx$; (C) $\int x e^{x^2} \, dx$; (D) $\int x \sqrt{x + 1} \, dx$.

(2) The derivative of $\frac{1}{x}$ is $\frac{1}{x^2}$.

(3) Find a function whose second derivative is $\frac{1}{x^2}$. Your answer: $-\ln x + C$.

(4) Which of the following is correct? Only one choice is correct.

(A) The antiderivative of $\frac{1}{x^2}$ is $\ln(x^2) + C$.

(B) The antiderivative of $e^{-x}$ is $e^{-x} + C$.

(C) The derivative of $\frac{x^2}{x^3 + 1}$ is $\frac{2x}{3x^2 + 1}$.

(D) Any two antiderivative functions of a fixed function $f(x)$ can only differ by a constant.

(5) In the figure on the right, the function $f(x)$ encloses several regions with $x$-axis. The area for the indicated regions is listed as follows:

- Region A Area: 9
- Region B Area: 6
- Region C Area: 5
- Region D Area: 3
- Region E Area: 2

Then the definite integral $\int_0^8 f(x) \, dx$ equals $5$.

Solution

(1) (B)

(2) Write $\frac{1}{x} = x^{-1}$. By the derivative power rule, we have its derivative $(-1) \cdot x^{-2} = -\frac{1}{x^2}$.

(3) If we need a function with second derivative $\frac{1}{x^2}$, we just integrate $\frac{1}{x^2}$ twice. Write $\frac{1}{x^2} = x^{-2}$. Integrate once. We have $-\frac{1}{x} + C$ for some constant $C$. Integrate this function again. we have $-\ln x + Cx + D$ for arbitrary constants $C$ and $D$. So any function of this form will be a right answer. For example, $-\ln x$, or $-\ln x + 2x + 3$.

(4) (D)

(5) By the geometric definition of definite integrals, the value of a definite integral is the net area of regions enclosed by the curve between bounds. In the figure, between $x = 0$ and $x = 8$, the function encloses four regions A, B, C and D. Notice regions B and D are below x-axis. Thus the value of the integral is

$$\int_0^8 f(x) \, dx = \text{Area}(A) - \text{Area}(B) + \text{Area}(C) - \text{Area}(D) = 9 - 6 + 5 - 3 = 5$$
**Bonus Problem** Compute the following integral \( \int \sqrt{1 + \sqrt{x}} \, dx \).

**Solution** Take substitution \( u = \sqrt{x} \). Instead of directly differentiating it, we rewrite the substitution as \( x = u^2 \) and differentiate this relation by \( u \) to obtain:

\[
x = u^2 \quad \Rightarrow \quad dx = 2u \, du
\]

Then the original integral can be transformed into

\[
\int \sqrt{1 + \sqrt{x}} \, dx = \int \sqrt{1 + u} \cdot 2u \, du
\]

Now we are going to use integration by parts. Take \( f(u) = 2u \) and \( g(u) = \sqrt{1 + u} = (1 + u)^{1/2} \). Then we have

\[
f'(u) = 2 \quad G(u) = \int (1 + u)^{1/2} \, du = \frac{2}{3}(1 + u)^{3/2}
\]

Thus the resulting above integral now equals:

\[
\int \sqrt{1 + u} \cdot 2u \, du = 2u \cdot \frac{2}{3}(1 + u)^{3/2} - \int 2 \cdot \frac{2}{3}(1 + u)^{3/2} \, du
\]

\[
= \frac{4}{3}u(1 + u)^{3/2} - \frac{4}{3} \int (1 + u)^{3/2} \, du
\]

\[
= \frac{4}{3}u(1 + u)^{3/2} - \frac{4}{3} \cdot \frac{2}{5}(1 + u)^{5/2} + C
\]

\[
= \frac{4}{3} \sqrt{x}(1 + \sqrt{x})^{3/2} - \frac{8}{15}(1 + \sqrt{x})^{5/2} + C
\]