### The first derivative test

1. **Critical points**: All solutions of the equation $f'(x) = 0$
   - Critical points will divide the real into subintervals.
   - Critical points contain all local maximal and minimal points.
   - Over each subinterval, the function $f(x)$ is all increasing or all decreasing.

2. **Test of increase/decrease on each subinterval**
   - Pick a representative number $x = a$ in each subinterval.
   - Compute the derivative value $f'(a)$ and check its sign.
   - If $f'(a) > 0$, then $f(x)$ is increasing on this subinterval.
   - If $f'(a) < 0$, then $f(x)$ is decreasing on this subinterval.

3. **Classification of Critical points**:
   - Maximum: \[ \uparrow \text{ max} \downarrow \]
   - Minimum: \[ \downarrow \text{ min} \uparrow \]

### The Second derivative test

1. **Possible inflections**: All solutions of the equation $f''(x) = 0$
   - Possible inflections will divide the real into subintervals.
   - Possible inflections contain all inflection points.
   - Over each subinterval, the function $f(x)$ is all concave up or all concave down.

2. **Test of concavity on each subinterval**
   - Pick a representative number $x = a$ in each subinterval.
   - Compute the second derivative value $f''(a)$ and check the sign.
   - If $f''(a) > 0$ then $f(x)$ is concave up on this subinterval.
   - If $f''(a) < 0$ then $f(x)$ is concave down on this subinterval.

3. **Classification of PIFs**:
   - Inflection point: points where $f(x)$ changes concavity.
Example. Sketch a rough graph of the function \( f(x) = \frac{1}{3}x^3 - x^2 - 3x + 1 \).

1. Compute \( f'(x) \): \( f'(x) = x^2 - 2x - 3 \)

2. Find critical points:
   Set \( f'(x) = x^2 - 2x - 3 = 0 \).
   \((x + 1)(x - 3) = 0 \Rightarrow x = -1 \) or \( x = 3 \)

3. First derivative test:

<table>
<thead>
<tr>
<th>Critical points</th>
<th>( x = -1 )</th>
<th>( x = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subintervals</td>
<td>( x &lt; -1 )</td>
<td>( -1 &lt; x &lt; 3 )</td>
</tr>
<tr>
<td>Rep. points</td>
<td>(-2)</td>
<td>(0)</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>(5 &gt; 0)</td>
<td>(-3 &lt; 0)</td>
</tr>
</tbody>
</table>

   Original \( f(x) \): \( \uparrow \) max \( \downarrow \) min \( \uparrow \)

4. Compute \( f''(x) \): \( f''(x) = 2x - 2 \)

5. Find possible inflection points:
   Set \( f''(x) = 2x - 2 = 0 \).
   \( 2x - 2 = 0 \Rightarrow x = 1 \)

6. Second derivative test:

<table>
<thead>
<tr>
<th>Critical points</th>
<th>( x = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subintervals</td>
<td>( x &lt; 1 )</td>
</tr>
<tr>
<td>Rep. points</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>( f''(x) )</td>
<td>(-2 &lt; 0)</td>
</tr>
</tbody>
</table>

   Original \( f(x) \): \( \cap \) IF \( \cup \)

7. Compute the \( y \)-value of all CPs and PIFs, using the original \( f(x) \)
   - At \( x = -1 \): \( y = \frac{1}{3} \cdot (-1)^3 - (-1)^2 - 3 \cdot (-1) + 1 = \frac{8}{3} \)
     \( \text{Point } (-1, \frac{8}{3}) \) Max
   - At \( x = 3 \): \( y = \frac{1}{3} \cdot 3^3 - 3^2 - 3 \cdot 3 + 1 = -8 \)
     \( \text{Point } (3, -8) \) Min
   - At \( x = 1 \): \( y = \frac{1}{3} \cdot 1^3 - 1^2 - 3 \cdot 1 + 1 = -\frac{8}{3} \)
     \( \text{Point } (1, -\frac{8}{3}) \) Inflection

8. Sketch the graph. The graph must show all features we obtained from both derivative tests.
Example. Sketch a rough graph of the function \( f(x) = 3x^4 - 8x^3 + 2 \).

1. Compute \( f'(x) \): \( f'(x) = 12x^3 - 24x^2 \)

2. Find critical points:
   Set \( f'(x) = 12x^3 - 24x^2 = 0 \).
   \[ 12x^2(x - 2) = 0 \Rightarrow x = 0 \text{ or } x = 2 \]

3. First derivative test:

<table>
<thead>
<tr>
<th>Critical points</th>
<th>( x = 0 )</th>
<th>( x = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subintervals</td>
<td>( x &lt; 0 )</td>
<td>( 0 &lt; x &lt; 2 )</td>
</tr>
<tr>
<td>Rep. points</td>
<td>( -1 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>(-36 &lt; 0)</td>
<td>(-12 &lt; 0)</td>
</tr>
<tr>
<td>Original ( f(x) )</td>
<td>( \searrow \text{ dec} )</td>
<td>( \searrow \text{ min} )</td>
</tr>
</tbody>
</table>

4. Compute \( f''(x) \): \( f''(x) = 24x^2 - 48x \)

5. Find possible inflection points:
   Set \( f''(x) = 36x^2 - 48x = 0 \).
   \[ x(36x - 48) = 0 \Rightarrow x = 0 \text{ or } x = \frac{4}{3} \]

6. Second derivative test:

<table>
<thead>
<tr>
<th>Critical points</th>
<th>( x = 0 )</th>
<th>( x = \frac{4}{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subintervals</td>
<td>( x &lt; 0 )</td>
<td>( 0 &lt; x &lt; \frac{4}{3} )</td>
</tr>
<tr>
<td>Rep. points</td>
<td>( x = -1 )</td>
<td>( x = 1 )</td>
</tr>
<tr>
<td>( f''(x) )</td>
<td>( 84 &gt; 0 )</td>
<td>(-12 &lt; 0 )</td>
</tr>
<tr>
<td>Original ( f(x) )</td>
<td>( \cup \text{ IF} )</td>
<td>( \cap \text{ IF} )</td>
</tr>
</tbody>
</table>

7. Compute the \( y \)-value of all CPs and PIFs, using the original \( f(x) \)

| At \( x = 0 \): | \( y = 3 \cdot 0^4 - 8 \cdot 0^3 + 2 = 2 \) | Point \((0, -2)\) Dec & Inflection |
| At \( x = 2 \): | \( y = 3 \cdot 2^4 - 8 \cdot 2^3 + 2 = -14 \) | Point \((2, -14)\) Min |
| At \( x = \frac{4}{3} \): | \( y = 3 \cdot \left(\frac{4}{3}\right)^4 - 8 \cdot \left(\frac{4}{3}\right)^3 + 2 = \frac{202}{27} \approx -7.48 \) | Point \((\frac{4}{3}, -7.48)\) Inflection |

8. Sketch the graph. The graph must show all features we obtained from both derivative tests.