1 (15 points) Find the derivative of the following functions. You don’t need to simplify your answer.

(1) \( y = \frac{2}{(x^3 + 2x)^{10}} \); \( y = \frac{1}{x^3} - x\sqrt{3-2x} \); \( y = \left( \frac{x^2 + 1}{x} \right)^3 \).

**Solution** (1) The function has a composition of the inside function \( u = x^3 + 2x \) and outside \( y = \frac{2}{u^{10}} = 2\cdot u^{-10} \). By the chain rule, we have that

\[
\frac{dy}{dx} = 2 \cdot (-10)u^{-11} \cdot (5x^4 + 2) = -20(x^5 + 2x)^{-11} \cdot (5x^4 + 2)
\]

(2) Decompose the function \( y \) into two components, \( \frac{1}{x^3} = x^{-3} \) and \( x\sqrt{3-2x} \). They are connected by subtraction. Then by the difference rule, we have

\[
y' = \left[ x^{-3} \right] - \left[ x\sqrt{3-2x} \right]
\]

(1)

Part (I) can be easily done by the power rule: \( I = [x^{-3}]' = -3x^{-4} \).

The function involved in Part (II) is a product function of factors \( x \) and \( \sqrt{3-2x} \). So we identify \( f(x) = x \) and \( g(x) = \sqrt{3-2x} \) and want to apply the product rule:

\[
f(x) = x \quad g(x) = \sqrt{3-2x} = (3-2x)^{1/2}
\]

\[
f'(x) = 1 \quad g'(x) = \frac{1}{2}(3-2x)^{-1/2} \cdot (-2) \quad (\text{by chain rule with } u = 3-2x)
\]

Thus part (II) by the product rule equals:

\[
(II) = \left[ x\sqrt{3-2x} \right]' = \frac{1}{f(x)} \sqrt{3-2x} + x \cdot \frac{1}{2}(3-2x)^{-1/2} \cdot (-2)
\]

Finally, put the above results in (1), the derivative of the original function is

\[
y' = (I) - (II) = -3x^{-4} - \sqrt{3-2x} - x \cdot \frac{1}{2}(3-2x)^{-1/2} \cdot (-2)
\]

(3) The original function has a composition of \( u = \frac{x^2 + 1}{x} \) and \( y = u^3 \). Then by the rule, the derivative is

\[
\frac{dy}{dx} = 3 \left( \frac{x^2 + 1}{x} \right)^2 \left[ \frac{x^2 + 1}{x} \right]' = 3 \left( \frac{x^2 + 1}{x} \right)^2 \left( \frac{2x}{x} - \frac{x^2 + 1 \cdot 1}{x^2} \right)
\]

(2)

To find the derivative function in part (I), we notice that the function involved is a quotient function. So we can identify

\[
f(x) = x^2 + 1 \quad g(x) = x
\]

\[
f'(x) = 2x \quad g'(x) = 1
\]

Then by the quotient rule, part (I) will equal \( I = \left[ \frac{x^2 + 1}{x} \right]' = \frac{2x \cdot x - (x^2 + 1) \cdot 1}{x^2} \)

Therefore, by the equation (2), the derivative of \( y \) is

\[
\frac{dy}{dx} = 3 \left( \frac{x^2 + 1}{x} \right)^2 \left( \frac{2x}{x} - \frac{x^2 + 1 \cdot 1}{x^2} \right) = 3 \left( \frac{x^2 + 1}{x} \right)^2 \frac{2x \cdot x - (x^2 + 1) \cdot 1}{x^2}
\]
2. (15 points) To approximate the decimal value of $\sqrt{5}$, we regard $\sqrt{5}$ as the solution of the equation $x^2 - 5 = 0$. Now choose the initial approximation $x_1 = 2$. Use the Newton's method to give the approximations $x_2$ and $x_3$.

**Solution** Set $f(x) = x^2 - 5$. Its derivative is $f'(x) = 2x$. Then the recursive formula for the Newton's method is

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}$$

Using the initial approximation $x_1 = 2$, we can recursively find the following approximations:

$$x_2 = x_1 - \frac{x_1^2 - 5}{2x_1} = 2 - \frac{4 - 5}{4} = 1.75$$

$$x_3 = x_2 - \frac{x_2^2 - 5}{2x_2} = 1.75 - \frac{1.75^2 - 5}{2 \cdot 1.75^2} = 1.71088$$

3. (15 points) Suppose we want to build a rectangular box with NO LID lid.

The box has a square base. Suppose the material for the bottom is $5$ per square foot and the material for the side is $4$ per square foot. Find the dimensions of this box of fixed volume $320$ feet$^3$ with the minimal total cost on the material.

**Solution**

(1) Objective function: We want to find the minimal cost to build this box. Thus the objective variable is the cost. This is an open box with no lid. We measure the cost for each face:

<table>
<thead>
<tr>
<th>Face</th>
<th>Area</th>
<th>Unit cost</th>
<th>Cost for this face</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom</td>
<td>$x^2$</td>
<td>5</td>
<td>$5x^2$</td>
</tr>
<tr>
<td>Left</td>
<td>$xy$</td>
<td>4</td>
<td>$4xy$</td>
</tr>
<tr>
<td>Right</td>
<td>$xy$</td>
<td>4</td>
<td>$4xy$</td>
</tr>
<tr>
<td>Front</td>
<td>$xy$</td>
<td>4</td>
<td>$4xy$</td>
</tr>
<tr>
<td>Back</td>
<td>$xy$</td>
<td>4</td>
<td>$4xy$</td>
</tr>
</tbody>
</table>

Thus the total cost of this box is the sum:

$$C = 5x^2 + 16xy$$

(2) The constraint equation: In this problem, our constraint is the volume condition. The volume of the box we want to build has to be $320$ feet$^3$. Since the volume of a rectangular box is the product of the length, the width and the height, the constraint equation is

$$x^2y = 320$$

(3) Solve the above constraint equation for the $y$-variable as a function of the $x$-variable:

$$x^2y = 320 \quad \Rightarrow \quad y = \frac{320}{x^2}$$

(4) Put the above relation in the objective function in part (1) to reduce the variable $y$:

$$C = 5x^2 + 16 \cdot \frac{320}{x^2} = 5x^2 + \frac{5120}{x}$$

(5) Take the derivative of the objective function and find the critical point:

$$C'(x) = 10x - \frac{5120}{x^2} = 0 \quad \Rightarrow \quad 10x^3 - 5120 = 0 \quad \Rightarrow \quad x^3 = 512 \quad \Rightarrow \quad x = 8$$

Thus we obtain a critical point $x = 8$. This is the potential $x$-dimension that will minimize or maximize the total cost $C$. 

(6) We need to verify that when \( x = 8 \), the total cost \( C \) is actually minimized. To do that, we apply the derivative test by picking representative numbers around the critical point \( x = 8 \):

<table>
<thead>
<tr>
<th>Critical points</th>
<th>( x = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rep. points</td>
<td>6</td>
</tr>
<tr>
<td>Derivative ( C'(x) = 10x - \frac{5120}{x^2} )</td>
<td>(-82.2 &lt; 0)</td>
</tr>
<tr>
<td>Original ( C(x) )</td>
<td>Dec</td>
</tr>
</tbody>
</table>

Thus at \( x = 8 \), we conclude the total cost of the box is the smallest.

(7) At \( x = 8 \) (feet), we use the relation in part (3) to get the \( y \)-dimension:

\[
y = \frac{320}{x^2} = \frac{320}{8^2} = 5 \text{ (feet)}
\]

And the minimal total cost is, by the objective function in part (1), is

\[
C = 5x^2 + 16xy = 5 \cdot 8^2 + 16 \cdot 8 \cdot 5 = 960 \text{ (dollars)}
\]

4 (15 points) (1) The following are the graphs of three functions, with a few points labeled.

![Graphs of three functions](image)

Fill out the following table regarding the local properties of the labeled points.

<table>
<thead>
<tr>
<th>Point</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of the first derivative</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sign of the second derivative</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Solution**

<table>
<thead>
<tr>
<th>Point</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of the first derivative</td>
<td>−</td>
<td>+</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sign of the second derivative</td>
<td>+</td>
<td>0</td>
<td>−</td>
<td>0</td>
</tr>
</tbody>
</table>
(2) Suppose there is a function \( f(x) \). Its first derivative \( f'(x) \) and second derivative \( f''(x) \) are shown below. Use the information of the first and second derivative function to give a possible graph of the function \( f(x) \).

\[
\begin{align*}
\text{Solution} & \quad \text{Read the graph of } f'(x), \text{ we see that } f(x) \text{ is increasing at } x < -1 \text{ and } x > 1, \text{ and is decreasing } f(x) \text{ at } -1 < x < 1. \\
& \quad \text{So } f(x) \text{ will achieve a local maximum at } x = -1 \text{ and a local minimum at } x = 1. \\
& \quad \text{Then Read the graph of } f''(x), \text{ we see that } f(x) \text{ is concave up at } x > 0 \text{ and concave down at } x < 0. \text{ So } f(x) \text{ has an inflection point at } x = 0. \\
& \quad \text{Then sketch a graph of } f(x) \text{ displaying all the above information.}
\end{align*}
\]

(3) Find the derivative of \( y = \sqrt{x^2 + \sqrt{x^2 + \sqrt{x^2}}} \).

\[
\text{Solution} \quad \text{Apply the chain rule twice:}
\]

First write \( y = (x^2 + \sqrt{x^2 + \sqrt{x^2}})^{1/2} \) and treat \( u = x^2 + \sqrt{x^2 + \sqrt{x^2}} \) and \( y = u^{1/2} \). Then by the chain rule, we have

\[
y' = \frac{1}{2} \cdot (x^2 + \sqrt{x^2 + \sqrt{x^2}})^{-1/2} \cdot \left[ x^2 + \sqrt{x^2 + \sqrt{x^2}} \right]' \\
= \frac{1}{2} \cdot (x^2 + \sqrt{x^2 + \sqrt{x^2}})^{-1/2} \cdot \left( 2x + \frac{1}{2} (x^2 + \sqrt{x^2 + \sqrt{x^2}})^{-1/2} \cdot 2x \right) \\
\]

For part (i), the derivative of \( \sqrt{x^2 + \sqrt{x^2}} = (x^2 + \sqrt{x^2})^{1/2} \). Take the inside function as \( x^2 + \sqrt{x^2} \) and apply the chain rule again:

\[
(i) = \frac{1}{2} \cdot (x^2 + \sqrt{x^2})^{-1/2} \cdot [x^2 + \sqrt{x^2}]' = \frac{1}{2} \cdot (x^2 + \sqrt{x^2})^{-1/2} \cdot (2x)
\]

Combine the above result and equation (3):

\[
y' = \frac{1}{2} \cdot (x^2 + \sqrt{x^2 + \sqrt{x^2}})^{-1/2} \cdot \left[ 2x + \frac{1}{2} (x^2 + \sqrt{x^2})^{-1/2} \cdot (2x) \right]
\]
**Bonus**  
(Bonus Problem. 10 points) Find the point on the parabola $f(x) = \sqrt{x}$ for $x \geq 0$ that is closest to the point $(2, 0)$. I.e., the distance between $(2, 0)$ and a point on the function $f(x)$ is the smallest.

**Diagram**

---

$f(x) = \sqrt{x}$

$(x, y)$

minimize the distance

$(2, 0)$

---

**Solution**  
The distance between the point $(2, 0)$ and $(x, y)$ is the objective function $d = \sqrt{(x-2)^2 + y^2}$. The constraint equation is $y = \sqrt{x}$. Put this relation into the objective:

$$d = \sqrt{(x-2)^2 + y^2} = \sqrt{x^2 - 3x + 4}$$

Find the critical point of $d$:

$$d'(x) = \frac{1}{2} \cdot \frac{2x - 3}{\sqrt{x^2 - 3x + 4}} \leq 0 \Rightarrow 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$$

Check that $x = \frac{3}{2}$ is a minimal point by applying the derivative test:

<table>
<thead>
<tr>
<th>Critical points</th>
<th>$x = \frac{3}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample points</td>
<td>1</td>
</tr>
<tr>
<td>First derivative $d'(x)$</td>
<td>$-$</td>
</tr>
<tr>
<td>Function $d(x)$</td>
<td>$\downarrow$ min $\uparrow$</td>
</tr>
</tbody>
</table>

This table shows that $x = \frac{3}{2}$ is actually the minimal point. Therefore the smallest distance will occur when $x = 1.5, y = \sqrt{x} = \sqrt{1.5}$ with shortest distance $d = \sqrt{1.5^2 - 3 \cdot 1.5 + 4} = \sqrt{1.75}$. 

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