Problems:

1. Determine if the functions shown in the graph are increasing/decreasing and concave up/down at all given points, and then determine the sign (+, 0 or −) of \( f'(x) \) and \( f''(x) \) at these points.

<table>
<thead>
<tr>
<th>Point</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
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<tbody>
<tr>
<td>Inc/Dec/Max/Min</td>
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<tr>
<td>Sign of ( f'(x) )</td>
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<tr>
<td>Concavity/Inflection</td>
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<td>Sign of ( f''(x) )</td>
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2. The figure below shows eight functions. Find the functions satisfying some specific conditions:

(a) (b) (c) (d) (e) (f) (g) (h)

(1) Which functions have a positive \( f'(x) \) for all \( x \)?
(2) Which functions have a negative \( f'(x) \) for all \( x \)?
(3) Which functions have a positive \( f''(x) \) for all \( x \)?
(4) Which functions have a negative \( f''(x) \) for all \( x \)?
(5) Which functions have local maximal points or local minimal points?
(6) Which functions have inflection points?
3. Suppose we have the results of the first and second derivative test about a function $f(x)$:

<table>
<thead>
<tr>
<th>Intervals</th>
<th>$x &lt; -2$</th>
<th>$-2 &lt; x &lt; 1$</th>
<th>$1 &lt; x &lt; 3$</th>
<th>$x &gt; 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Intervals</th>
<th>$x &lt; 0$</th>
<th>$0 &lt; x &lt; 2$</th>
<th>$2 &lt; x &lt; 5$</th>
<th>$x &gt; 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f''(x)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

Also assume we know the function values at some integer points:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$-2$</td>
<td>$0$</td>
<td>$3$</td>
<td>$1$</td>
<td>$0$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

(1) Write down the intervals where the function is increasing or decreasing. Find all local extremes.
(2) Write down the intervals where the function is concave up or concave down. Find all inflection points.
(3) Plot all typical points on a $xy$–plane and connect them to a rough graph of $f(x)$:

4. Given a function $f(x) = x^3 + 6x^2 + 9x$, do the following:
(1) Find critical points of $f(x)$;
(2) Apply the first derivative test to determine the increasing/decreasing property and local extremes of $f(x)$;
(3) Find all possible inflection points of $f(x)$;
(4) Apply the second derivative test to determine the concavity and inflection points of $f(x)$;
(5) Sketch the graph of $f(x)$ including all typical points.

5. Use the derivative tests to sketch the graph of $f(x) = -x^3 + 3x^2 + 1$.

6. Use the derivative tests to sketch the graph of $f(x) = x^4 - 6x^2$.

7. Use the derivative tests to sketch the graph of $f(x) = -x^4 + 4x^3$.

8. Each of the figures shown below has two functions $f(x)$ and $g(x)$. Determine which function is the derivative of the other.

(1) ![Graph 1](image1.png)  
(2) ![Graph 2](image2.png)
9. Find two positive numbers such that their sum is 40 and their product is maximal.

10. Given a piece of rope of length 20 centimeters, find the rectangle of largest area that it can enclose.

11. (Optional.) A unit circle on the \(xy\)-plane is a circle centered at the origin with radius 1. This unit circle can be written as an equation \(x^2 + y^2 = 1\). That is, any point on this unit circle must satisfy this equation. Now find the dimensions of the rectangle of maximum area that can be inscribed in the unit circle, in the way shown in the figure.

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**Homework 7 Answer**

1. 

<table>
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<tr>
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<td>Max</td>
<td>Dec</td>
<td>Min</td>
<td>Inc</td>
<td>Dec</td>
<td>Dec</td>
<td>Dec</td>
</tr>
<tr>
<td>Sign of (f'(x))</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Concavity/Inflection</td>
<td>Down</td>
<td>Inf</td>
<td>Up</td>
<td>Up</td>
<td>Down</td>
<td>Inf</td>
<td>Up</td>
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<tr>
<td>Sign of (f''(x))</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>0</td>
<td>+</td>
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</table>

2. (1) c, e, h; (2) b, f; (3) a, b, c; (4) f; (5) a, d, g; (6) d, e, g, h.

3. (1) Increasing intervals: \(-2 < x < 1\) and \(x > 3\); decreasing intervals: \(x < -2\) and \(1 < x < 3\). Local maximum: \((1,3)\); Local minima: \((-2,-2)\) and \((3,0)\).

   (2) Concave up: \(x < 0\) and \(2 < x < 5\); Concave down: \(0 < x < 2\) and \(x > 5\). Inflection points: \((0,0),(2,1)\) and \((5,2)\).

   (3)
4. (1) \((-3,0),(-1,-4)\);
(2) Increasing on \(x < -3\) and \(x > -1\), decreasing on \(-3 < x < -1\);
(3) \((-2,-2)\);
(4) Concave up on \(x > -2\), concave down on \(x < -2\); (5) Graph: on the right.

5. See the graph on the left below.
Local max: \((2,5)\); local min: \((0,1)\); inflection: \((1,3)\).

6. See the graph in the middle below.
Local max: \((0,0)\); local min: \((-\sqrt{3},9),(\sqrt{3},9)\); inflection: \((1,-5),(-1,-5)\).

7. See the graph on the right below.
Local max: \((6,13.5)\); local min: None; inflection: \((0,0)\) and \((4,8)\). Note that \((0,0)\) is also a critical point, but it is neither a maximum nor a minimum.

8. (1) \(f(x) = g'(x)\); (2) \(g(x) = f'(x)\).

9. Setup: let \(x, y\) be two positive numbers; Goal: minimize \(x + y\); Constraint: \(xy = 36\). Reduced function \(f(x) = x + \frac{36}{x}\). Minimal point: \(x = 6, y = 6\) and the minimal sum is 12.

10. Setup: let \(x, y\) be the dimensions of this rectangle. Goal: maximize \(xy\); Constraint: \(2x + 2y = 20\); Reduced function \(f(x) = x(10 - x) = 10x - x^2\). Maximal point: \(x = 5, y = 5\) and maximal area is 25.

11. Setup: let \(x, y\) be a point on the unit circle in the first quadrant. Goal: maximize \(4xy\); Constraint: \(x^2 + y^2 = 1\); Reduced function \(f(x) = 4x\sqrt{1 - x^2}\). Derivative \(f'(x) = \frac{-4x^2}{\sqrt{1 - x^2}} + 4\sqrt{1 - x^2} = \frac{4 - 8x^2}{\sqrt{1 - x^2}}\). Maximal point: \(x = 1/\sqrt{2}, y = 1/\sqrt{2}\) and maximal area is 2.