Problems:

1. (1) Suppose we know a relation between two variables $x$ and $y$ is $\ln\left(\frac{2}{y}\right) + x^2 = 2x + 1$. Can you isolate $y$ and write $y$ as a function of $x$?

(2) Suppose we know a relation between two variables $x$ and $y$ is $x^2e^{\sqrt{y} + 1} = 2x + 1$. Can you isolate $y$ and write $y$ as a function of $x$?

2. Differentiate the following functions:
   (1) $y = \ln(x^4 + 3x^3 + 1)$; (chain rule) (2) $y = \frac{x}{\ln x}$; (quotient rule)
   (3) $y = e^{\sqrt{x}}$; (chain rule) (4) $y = xe^{(x^2)}$; (product rule and chain rule)
   (5) $y = (\sqrt{x} + 1)e^{-2x}$; (product rule and chain rule) (6) $y = \ln(\ln x)$; (chain rule)

3. Differentiate the following functions:
   (1) $y = 3(x^4 + 1)^5 - 4e^{2x}$; (2) $y = \sqrt{\ln(x^2 + 1)} - \frac{1}{e^x}$;
   (3) $y = \frac{e^{2x} + 1}{x - 1} - \frac{1}{(\ln x)^2}$; (4) (Hard) $y = \ln\left(\frac{x^2 + 1}{2x + 3} + \left(\frac{3x^2 + 3}{e^{4x}}\right)^2\right)$; (simplify the function first.)

4. Calculate the first, second and third derivative of $f(x) = e^{x^2 + 1}$ and evaluate them at $x = 0$.

5. Calculate the first, second and third derivative of $f(x) = x^3\ln x$ and evaluate them at $x = 1$.

6. Use the method of logarithmic differentiation to compute the derivative of
   (1) $f(x) = (x + 1)^4(4x - 1)^2$; (2) $f(x) = e^{2x}(3x - 4)^6$;
   (3) $f(x) = \frac{(x + 1)^2(x^4 + 2x + 1)}{\sqrt{3x + 1}}$; (4) (Hard) $f(x) = \sqrt{\frac{(x^2 + 1)(x^2 + 2x + 3)^2}{e^{x^2 + 1}}}$;

7. Let $P(t)$ be the population (in millions) of a certain city $t$ years after 2010. (That is, the year 1990 corresponds to $t = 0$.) Suppose in 2010 the population was 3 million and the growth constant is 2%.
   (1) Find the formula of $P(t)$;
   (2) What was the population in 2012;
   (3) In which year will the population reach 4 million?

8. The size of a certain insect population is given by $P(t) = 300e^{0.1t}$ where $t$ is measured in days.
   (1) How many insects were present initially?
   (2) Calculate the population after 3 days.
   (3) At what time will the population double?
   (4) At what time will the population equal 1200?
9. Ten grams of a radioactive substance with decay constant 0.04 is stored in a vault. Assume that time is measured in days, and let \( P(t) \) be the amount remaining at time \( t \).

(1) Give a formula of \( P(t) \);
(2) How much will remain after 5 days?
(3) What is the half-life of this radioactive substance?

10. Ten grams of a radioactive material disintegrates to 3 grams in 5 years. What is the half-life of the radioactive material?

Homework 11 Answer

1. (1) \( y = \frac{2}{e^{-x^2+2x+1}} \); (2) \( y = \left( \frac{2x + 1}{x^2} \right)^2 - 1; \)

2. (1) \( y' = \frac{4x^2 + 9x^2}{x^2 + 3x + 1 + 1}; \) (2) \( y' = \frac{ln(x - 1)}{(lnx)^2}; \) (3) \( y' = \frac{1}{2}x^{-1/2} \cdot e^{y}; \)
(4) \( y' = e^{x^2} + 2x^2e^{x^2}; \) (5) \( y' = \frac{1}{2}x^{-1/2} \cdot e^{-2x} + (\sqrt{x} + 1) \cdot (-2)e^{-2x}; \) (6) \( y' = \frac{1}{x \ln x}; \)

3. (1) \( y' = 15x^4 + 1 \cdot 4x^2 - 8x^2; \) (2) \( y' = \frac{1}{2}(\ln(x^2 + 1))^{-1/2} \cdot \frac{2x}{x^2 + 1} - (-2)e^{-2x}. \)
(3) \( y' = \frac{2x^2(x - 1) - (x^2 + 1)}{(x - 1)^2} - (-2) \cdot (\ln x)^{-3} \cdot \frac{1}{x}; \) (4) \( y' = \frac{1}{2} \cdot \frac{2x^3}{x^2 + 1} \cdot \left( \frac{(2x + 3)(x^2 + 1) - 2}{(x + 3)^2} \right) + \frac{2(3x^2 + 3)6x^3 \cdot e^{y} - (3x^2 + 3)^2 \cdot 8e^y}{e^{ln}}; \)

4. \( f'(x) = 2xe^{x^2+1}; \) \( f''(x) = 2e^{x^2+1} + 2x \cdot 2xe^{x^2+1} = (4x^2 + 2)e^{x^2+1}; \) \( f'''(x) = 8x \cdot e^{x^2+1} + (4x^2 + 2) \cdot 2xe^{x^2+1}; \)
At \( x = 0, f'(0) = 0, f''(0) = 2e \) and \( f'''(0) = 0. \)

5. \( f'(x) = 3x^2 \ln x + x^2; \) \( f''(x) = 6x \ln x + 5x; \) \( f'''(x) = 6x \ln 11 + 11; \) At \( x = 1, f'(1) = 1, f''(1) = 5 \) and \( f'''(x) = 11. \)

6. (1) \( f'(x) = (x + 1)^3(4x - 1)^2 \cdot \left( \frac{4}{x + 1} + \frac{8}{4x - 1} \right); \) (2) \( f'(x) = e^{2x} (3x - 4)^8 \cdot \left( 2 + \frac{24}{3x - 4} \right); \)
(3) \( f'(x) = \frac{(x + 1)^2(x^2 + 2x + 1)}{\sqrt{2x + 1}} \cdot \left( \frac{2}{x + 1} + \frac{4x^2 + 2}{4x^2 + 2x - 1} - \frac{1}{2} \cdot \frac{3}{3x + 1} \right); \)
(4) \( f'(x) = \sqrt{x^5 + 1(x^2 + 2x + 3)^2} \cdot \frac{1}{x} \cdot \left( \frac{5x^4}{x^5 + 1} + \frac{2(2x + 2)}{x^2 + 2x + 3} - 2x \right); \)

7. (1) \( P(t) = 3e^{0.02t}; \) (2) 3.12 million; (3) After 14.4 years. That is in 2025.
8. (1) 300; (2) 405; (3) 69.3 days; (4) 138.6 days.
9. (1) \( P(t) = 10e^{-0.04t}; \) (2) 8.2 grams; (3) 17.3 days.

10. The decay constant is \(-0.24\). The half-life is 2.89 years.