Fall 2014 MTH122 Survey of Calculus and its Applications II
Homework 6 (for lectures on 9/30,10/2)
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Topics in this homework:
Topic 1 Graphing solutions for autonomous equations
1. Definition of autonomous equations.
2. Know how to find and interpret constant solutions.
3. Describe the shape of solution curves given any initial condition. Know how to find the long-term limit.

Topic 2 Separable equations
1. Definition of separable equations.
2. Know the standard steps to solve a separable equation.

Problems:

1. Check that \( y = (\frac{1}{3} t^{3/2} + 2)^2 \) is a solution to differential equation \( y' = \sqrt{ty} \) and initial condition \( y(0) = 4 \).

2. Which of the following differential equations are autonomous? (More than one correct answer.)
   (A) \( y' = -y + 2 \); (B) \( y' = t^2 y \); (C) \( y' + e^y = 0 \); (D) \( y' - t = y \); (E) \( y' + y^2 = 1 \).

3. Given a differential equation \( y' = y - 2 \). Do the following:
   (1) Find all constant solutions and draw them on a \( t \ y \) plane.
   (2) Given \( y(0) = 3 \), what’s the initial derivative value? Is \( y(t) \) going to increase or decrease? Draw a solution curve with initial condition \( y(0) = 3 \).
   (3) Given \( y(0) = 1 \), what’s the initial derivative value? Is \( y(t) \) going to increase or decrease? Draw a solution curve with initial condition \( y(0) = 1 \).
   (4) Draw solution curves for initial conditions \( y(0) = 0 \), \( y(0) = 1.5 \) and \( y(0) = 2.5 \).
   (5) If \( y(t) \) starts with \( y(0) = 2.5 \), what’s the limit of \( y(t) \) when \( t \to +\infty \)?
   (6) If \( y(t) \) starts with \( y(0) = 1.5 \), what’s the limit of \( y(t) \) when \( t \to +\infty \)?
4. Given a differential equation \( y' = y(2 - y) \). Do the following:

(1) Find all constant solutions and draw them on a \( t \)-\( y \)-plane.

(2) Given \( y(0) = 3 \), what's the initial derivative value? Is \( y(t) \) going to increase or decrease? Draw a solution curve with initial condition \( y(0) = 3 \).

(3) Given \( y(0) = -1 \), what's the initial derivative value? Is \( y(t) \) going to increase or decrease? Draw a solution curve with initial condition \( y(0) = -1 \).

(4) Given \( y(0) = 1 \), what's the initial derivative value? Is \( y(t) \) going to increase or decrease? Draw a solution curve with initial condition \( y(0) = 1 \).

(5) Draw more solution curves for initial conditions \( y(0) = -0.5, y(0) = 0.5, y(0) = 1.5 \) and \( y(0) = 2.5 \).

(6) If \( y(t) \) starts with \( y(0) = -0.5 \), what's the limit of \( y(t) \) when \( t \to +\infty \)?

(7) If \( y(t) \) starts with \( y(0) = 1.5 \), what's the limit of \( y(t) \) when \( t \to +\infty \)?

5. Suppose we have a differential equation \( y' = y(4 - y^2) \). The graph of the function \( f(y) = y(4 - y^2) \) is shown here:

(1) Find constant solutions to this differential equation, and draw them on a \( t \)-\( y \)-plane.

(2) Draw solution curves on the plane for initial conditions \( y(0) = -3, y(0) = -1, y(0) = 1, \) and \( y(0) = 3 \).

(3) If \( y(t) \) starts with \( y(0) = 1.7 \), what's the limit of \( y(t) \) when \( t \to +\infty \)?

(4) If \( y(t) \) starts with \( y(0) = -0.6 \), what's the limit of \( y(t) \) when \( t \to +\infty \)?
6. Which of the following equations are separable? (More than one correct answer.)
(A) \( y' = 2y^2e^t \); (B) \( y' = t^3y + 2y \); (C) \( t^2y' = \ln y \);
(D) \( y' = t^2 + y^2 \); (E) \( y' = \cos y + 1 \); (F) \( y + y' = 3ty^2 \).

7. Follow the steps to solve the differential equation: \( y' = \frac{t + 5}{y^2} \) with initial condition \( y(0) = 1 \).
(1) Is this equation separable? Identify \( p(t) \) and \( q(y) \):

\[
p(t) = \quad q(y) =
\]

(2) Separation of variables: Notice that \( y' = \frac{dy}{dt} \). Now move \( q(y) \) to the left and move \( dt \) to the right:

\[
\quad dy = \quad dt
\]

(3) Make the above equality into integrals:

\[
\int \quad dy = \int \quad dt
\]

(4) Evaluate the integrals on two sides. Note that the integral on the left is an integral of \( y \) and the one on the right uses variable \( t \).

\[
\quad = \quad + C
\]

The above form is called the implicit solution.

(5) Solve the above implicit solution for an explicit \( y \):

\[
y = \quad
\]

The above form is called the general solution. Notice that the arbitrary constant \( C \) is included in the general solution.

(6) Now look at the initial condition \( y(0) = 1 \). Plug in \( t = 0 \) to the general solution in step (5) and derive an equation about \( C \):

\[
\quad = \quad
\]

and so the value of \( C \) is : \( C = \quad \).

Therefore the final unique solution is

\[
y = \quad
\]
Homework 6 Solution

1. Just check two sides of the equation are the same when we use \( y = (\frac{1}{3}t^{3/2} + 2)^2 \). Also check this function satisfies the initial condition.

2. (A),(C),(E)

3. (1) Constant solution: \( y = 2 \).
(2) Initial derivative: \( y'(0) = 1 \). The solution is increasing. See the blue curve on top.
(3) Initial derivative: \( y'(0) = -1 \). The solution is decreasing. See the blue curve at bottom.
(4) See all purple curves.
(5) Limit is \( +\infty \).
(6) Limit is \( -\infty \).

4. (1) Constant solutions : \( y = 0 \) and \( y = 2 \).
(2) Initial derivative: \( y'(0) = -3 \). The solution is decreasing. See the blue curve on top.
(3) Initial derivative: \( y'(0) = -3 \). The solution is decreasing. See the blue curve at bottom.
(4) Initial derivative: \( y'(0) = 1 \). The solution is increasing. See the blue curve in the middle.
(5) See all purple curves.
(6) Limit is \( -\infty \).
(7) Limit is 2.
5. (1) Constant solutions: $y = 0$, $y = 2$ and $y = -2$.
   (2) See the blue curves.
   (3) The limit is 2.
   (4) The limit is $-2$.

6. (A), (B), (C), (E).

7. (1) $p(t) = t + 5$, $q(y) = \frac{1}{y^2}$.
   (2) $y^2 \, dy = (t + 5) \, dt$.
   (3) $\int y^2 \, dy = \int (t + 5) \, dt$.
   (4) $\frac{y^3}{3} = \frac{t^2}{2} + 5t + C$.
   (5) $y = \sqrt[3]{\frac{3t^2}{2} + 15t + C}$.
   (6) Equation $\sqrt[3]{\frac{3t^2}{2} + 15 \cdot 0 + C} = 1$. Solution of $C$: $C = 1$. Final unique solution: $y = \sqrt[3]{\frac{3t^2}{2} + 15t + 1}$. 