Topics in this homework:

Topic 1. Partial derivatives

Topic 2. Maxima and minima of a multivariable function.

Problems:

1. Given that \( z = 3x^2 + 2xy + 5y \),
   (1) Find the (first) partial derivatives of \( z \): \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \);
   (2) Compute \( \frac{\partial z}{\partial x} (1, 4) \) and \( \frac{\partial z}{\partial y} (1, 4) \);
   (3) Find the second partial derivatives of \( z \): namely \( \frac{\partial^2 z}{\partial x^2} \), \( \frac{\partial^2 z}{\partial y^2} \), \( \frac{\partial^2 z}{\partial y \partial x} \) and \( \frac{\partial^2 z}{\partial x \partial y} \);
   (4) Check that \( \frac{\partial^2 z}{\partial y \partial x} \) and \( \frac{\partial^2 z}{\partial x \partial y} \) are the same;

2. Given that \( z = \sin(xy^2) \),
   (1) Find the (first) partial derivatives of \( z \): \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \);
   (2) Find the second partial derivatives of \( z \): namely \( \frac{\partial^2 z}{\partial x^2} \), \( \frac{\partial^2 z}{\partial y^2} \), \( \frac{\partial^2 z}{\partial y \partial x} \) and \( \frac{\partial^2 z}{\partial x \partial y} \);
   (3) Check that \( \frac{\partial^2 z}{\partial y \partial x} \) and \( \frac{\partial^2 z}{\partial x \partial y} \) are the same;

3. Find the first partial derivatives of the following functions:
   (1) \( z = \ln(\sqrt{x} + 2y^2) \); (2) \( z = \frac{e^{2x}}{1 - e^x} \) (use quotient rule);

4. This problem will compute all the relative extremes for the function \( z = x^3 + y^2 - 3x + 6y \).
   (1) Compute the partial derivatives of \( z \):
      \[
      \frac{\partial z}{\partial x} = \quad \frac{\partial z}{\partial y} =
      \]
   (2) Find all critical points of \( z \) by setting the above partial derivatives to zero. (Hint: we have two critical points in this function.)
      Critical points: __________________________
   (3) Compute the second partial derivatives:
      \[
      \frac{\partial^2 z}{\partial x^2} = \quad \frac{\partial^2 z}{\partial y^2} = \quad \frac{\partial^2 z}{\partial x \partial y} =
      \]
(4) Fill in the following table and determine if each critical point is a maximum, minimum, saddle or indeterminant.

<table>
<thead>
<tr>
<th>critical point</th>
<th>z</th>
<th>( \frac{\partial^2 z}{\partial x^2} )</th>
<th>( \frac{\partial^2 z}{\partial y^2} )</th>
<th>( \frac{\partial^2 z}{\partial x \partial y} )</th>
<th>Discriminant D</th>
<th>Type of extreme</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, -3)</td>
<td>-11</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>12</td>
<td>Minimum</td>
</tr>
<tr>
<td>(-1, -3)</td>
<td>-7</td>
<td>-6</td>
<td>2</td>
<td>0</td>
<td>-12</td>
<td>Saddle</td>
</tr>
</tbody>
</table>

5. Find the relative extremes for the following function: \( z = x^2 - 2xy + 4y^2 \).

6. Find the relative extremes for the following function: \( z = -3x^2 + 6xy - y^3 + 9y \).

Solution to Homework 12

1. (1) \( \frac{\partial z}{\partial x} = 6x + 2y \) and \( \frac{\partial z}{\partial y} = 2x + 5 \);
   (2) 14 and 7;
   (3) \( \frac{\partial^2 z}{\partial x^2} = 6, \frac{\partial^2 z}{\partial y^2} = 0, \frac{\partial^2 z}{\partial x \partial y} = 2 \);

2. (1) \( \frac{\partial z}{\partial x} = \cos(xy^2) \cdot y^2 \) and \( \frac{\partial z}{\partial y} = \cos(xy^2) \cdot 2xy \);
   (2) \( \frac{\partial^2 z}{\partial x^2} = -\sin(xy^2) \cdot y^4, \frac{\partial^2 z}{\partial y^2} = (-\sin(xy^2) \cdot 2xy) \cdot 2xy + \cos(xy^2) \cdot 2x, \)
   \( \frac{\partial^2 z}{\partial x \partial y} = (-\sin(xy^2) \cdot 2xy) \cdot y^2 + \cos(xy^2) \cdot 2y \) and \( \frac{\partial^2 z}{\partial y \partial x} = (-\sin(xy^2) \cdot y^2) \cdot 2xy + \cos(xy^2) \cdot 2y \);

3. (1) \( \frac{\partial z}{\partial x} = \frac{1}{2}x^{1/2} \) and \( \frac{\partial z}{\partial y} = \frac{4y}{\sqrt{x} + 2y^2} \);
   (2) \( \frac{\partial z}{\partial x} = \frac{2e^{2x} \cdot (1 - e^{xy}) + e^{2x} \cdot (-e^{xy} \cdot y)}{(1 - e^{xy})^2} \) and \( \frac{\partial z}{\partial y} = \frac{e^{2x} \cdot (-e^{xy} \cdot x)}{(1 - e^{xy})^2} \);

4. (1) \( \frac{\partial z}{\partial x} = 3x^2 - 3, \frac{\partial z}{\partial y} = 2y + 6 \);
   (2) Critical points: (1, -3), (-1, -3);
   (3) \( \frac{\partial^2 z}{\partial x^2} = 6x, \frac{\partial^2 z}{\partial y^2} = 2, \frac{\partial^2 z}{\partial x \partial y} = 0 \)

5. Relative minimum achieved at (0, 0) with value \( z = 0 \).
6. Relative maximum at (3, 3) with value 27 and a saddle point at (-1, -1) with value -5.