Problem 1. (20 points)
Given an arbitrary rectangular box with dimensions width $x$, depth $y$, and height $z$, as in the figure 1.

1. Let $V$ be the function of the volume of this box. Write $V$ as a function of $x, y, z$;

$$V = \quad \text{________________________}$$

2. Suppose we want to construct an open box (with no lid) like this. The cost for the base is $5 per unit of area, and for the four side faces the cost is $4 per unit of area. Let $C$ be the function of the total cost to construct this box.

$$C = \quad \text{________________________}$$

Problem 2. (20 points)
Suppose a double-variable function $z$ is defined as

$$z = e^{xy} + \sin(xy)$$

1. Compute the first derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$;

2. Compute the second derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, and $\frac{\partial^2 z}{\partial x \partial y}$.

Problem 3. (20 points)
Given a function

$$z = 3x^3y + y^3 - 9x^2y + 10$$

1. Find all critical points;

2. Use second derivative tests to determine the extreme type of each of the critical point. (Hint: If discriminant $D$ is 0, then the type is inconclusive. )

Problem 4. (20 points)
(Note. Problem updated 11/24: I used to use $f(x, y) = 5000 - (6x^2 + y^2)$ but actually the constant term should be 10000. The new function will match the solution then.)

(Method of Lagrange multiplier) The amount of revenue earned by a particular firm is

$$f(x, y) = 10000 - (6x^2 + y^2)$$

where $x$ and $y$ are, respectively, the number of units of labor and capital utilized. Suppose that the labor costs $30 per unit and capital costs $10 per unit and that the firm as 1000 to spend. Determine the amounts of labor and capital that should be utilized in order to maximize the revenue.
Problem 5. The following are some questions of medium difficulty. Fill the blank with the final answer. You don’t need to show your computation. (20 points)

1. Suppose $z = \frac{x\sqrt{y}}{1+x}$. Treat this function as a surface over $x y$-plane. Compute the height of point with $x = 2, y = 9$. Answer: ___

2. Suppose $z = f(x, y) = x^{100} y^{100}$. Find $\frac{\partial^2 z}{\partial x \partial y}$ at $(1, 1)$. Answer: ___

3. Suppose $z = 2x^2 + y^3 - xy^2 - 12y + 7$. Find the trace curve if we keep $x$ to be equal to 2: $z = ___$

4. Find the maximal product $x y$ for two positive real numbers under the condition that the sum $x + 2y = 32$. The maximal product is ___

Problem 6. The following are some short answer questions which might be challenging. (20 points)

1. (HARD) The critical point of the sphere $z = -\sqrt{36 - (x-2)^2 - (y-1)^2}$ is $x = _____, y = _____$ and $z = _____$.

2. (HARD) A partial differential equation, by definition, is an equation involving partial derivatives. Assume we know that $z = \sin(2x)e^{ay}$ is a solution to the equation $\frac{\partial^2 z}{\partial x^2} = 2 \frac{\partial z}{\partial y}$, where $A$ is an unknown parameter. Find value of $A$: ___

3. (HARD) If $w = f(x, y, z)$ is a function of three independent variables $x, y, z$, such as the volume function in the first problem, we can compute the partial derivatives in a similar way. If we want to differentiate $w$ with respect to some independent variable, we only need to do the regular differentiation by treating the other two variables as constants. I.e., $\frac{\partial w}{\partial x}$ is the regular derivative by $x$ when we treat $y$ and $z$ as constants. For example, if $w = f(x, y, z) = x^2 yz^2$, then $\frac{\partial w}{\partial x} = 2xyz^2$.

Now compute the partial derivative $\frac{\partial w}{\partial x}$ for the triple-variable function $w = \sin(\frac{2 \pi}{z})$. Answer: ___

4. (HARD) Suppose a double variable function $z$ has the form $z = f(x) + g(y) + c$, where $f(x)$ is a function of ONLY $x$ and $g(y)$ is a function of ONLY $y$, and $c$ is an arbitrary constant. Then computing partial derivatives gives $\frac{\partial z}{\partial x} = f'(x)$ and $\frac{\partial z}{\partial y} = g'(y)$. For instance, if $z = x^3 + y^4 + 5$. If we identify $f(x) = x^3$ and $g(y) = y^4$, we can check by calculation that $f'(x) = 3x^2$ which is also the partial derivative $\frac{\partial z}{\partial x}$, and $g'(y) = 4y^3$ which is also the partial derivative $\frac{\partial z}{\partial y}$.

Now we want to do the opposite. Suppose we know that $\frac{\partial z}{\partial x} = 3x^2, \frac{\partial z}{\partial y} = 4y$ and $z = 20$ when $x = 2, y = 3$. Try to recover the expression of this function: $z = ___$
Solution to practice exam 3

1. (1) $V = xyz$; (2) $C = 5xy + 8yz + 8xz$.

2. (1) \[ \frac{\partial z}{\partial x} = ye^{xy} + y \cos(xy); \quad \frac{\partial z}{\partial y} = xe^{xy} + x \cos(xy); \]
   (2) \[ \frac{\partial^2 z}{\partial x^2} = y^2e^{xy} - y^2 \sin(xy); \quad \frac{\partial^2 z}{\partial y^2} = x^2e^{xy} - x^2 \sin(xy); \]
   \[ \frac{\partial^2 z}{\partial x \partial y} = e^{xy} + xy e^{xy} + \cos(xy) - xy \sin(xy). \] (Use product rule.)

3. (1) $(0,0), (2,2), (2,-2), (3,0)$; (2) $(0,0)$ is inconclusive; $(2,2)$ is a minimum; $(2,-2)$ is a maximum; $(3,0)$ is a saddle.

4. Objective: maximize $f(x, y) = 10000 - (6x^2 + y^2)$; Constraint $g(x, y) = 30x + 10y = 1000$. By method of Lagrange multiplier, the only critical point is $x = 20$ and $y = 40$. In this case, the profit is $f(20, 40) = 6000$. Pick another pair, for instance, $(10, 70)$ such that $30x + 10y = 1000$, we check that $f(10, 70) = 4500$ which is smaller than 6000. So $x = 20, y = 40$ the maximal point with the maximal profit 6000.

5. (1) $2$; (2) $10000$; (3) $y^3 - 2y^2 - 12y + 15$; (4) $128$.

6. (1) $x = 2$, $y = 1, z = -6$; (2) $A = -2$; (3) $x \cdot \cos(\frac{1x}{2})$; (Solution updated 11/24) (4) $z = x^3 + 2y^2 - 6$. 