First 15 digits of Euler’s number: $e = 2.718281828459045$

Consider the differential equation $y’=y$ with IC $y(0)=1$. The unique solution is $y(t) = e^t$. So we have that $e=y(1)$.

Using Euler’s method, we can simply try to approximate the solution of the above differential equation at $t=1$.

Euler’s method: Approximation of the solution at $t=b$ from $t=0$ using $n$ steps:

$$
\text{Step size } h = \frac{b-0}{n}.
$$

Iterative formula:

$$
y_k = y_{k-1} + h 
= y_{k-1} + h \ast f(t_{k-1}, y_{k-1});$$

Approximation for $n=5$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$t_k$</th>
<th>$f(t_{k-1}, y_{k-1})$</th>
<th>$f(t_{k-1}, y_{k-1}) \ast h$</th>
<th>$y_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>0</td>
<td>---</td>
<td>---</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.2</td>
<td>1</td>
<td>0.2</td>
<td>1.2</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.4</td>
<td>1.2</td>
<td>0.24</td>
<td>1.44</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.6</td>
<td>1.44</td>
<td>0.288</td>
<td>1.728</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>0.8</td>
<td>1.728</td>
<td>0.3456</td>
<td>2.0736</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
<td>1</td>
<td>2.0736</td>
<td>0.41472</td>
<td>2.48832</td>
</tr>
</tbody>
</table>

Approximation of $e=y(1)$ 2.48832

Approximation using different number of steps

<table>
<thead>
<tr>
<th>number of steps</th>
<th>Approximation of $e$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.488320000000000</td>
<td>0.22996182845905</td>
</tr>
<tr>
<td>10</td>
<td>2.59374246010000</td>
<td>0.12453936835905</td>
</tr>
<tr>
<td>100</td>
<td>2.70481382942153</td>
<td>0.01346799903753</td>
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<tr>
<td>1,000</td>
<td>2.71692393223589</td>
<td>0.00135789622315</td>
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<tr>
<td>10,000</td>
<td>2.71814592682522</td>
<td>0.00013590163382</td>
</tr>
<tr>
<td>100,000</td>
<td>2.71826823717449</td>
<td>0.000001359128456</td>
</tr>
<tr>
<td>1,000,000</td>
<td>2.71828046931938</td>
<td>0.000000135913967</td>
</tr>
<tr>
<td>10,000,000</td>
<td>2.71828169254497</td>
<td>0.000000013591408</td>
</tr>
</tbody>
</table>

**Observation:** When the number of steps goes larger, the approximation will get better (i.e. closer to the real value of $e$).