Quiz 5 Solution

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Problem: The solution to the system of differential equations

\[
\begin{align*}
x_1' &= -3x_1 - 4x_3 \\
x_2' &= -x_1 - x_2 - x_3 \\
x_3' &= x_1 + x_3
\end{align*}
\]

with the initial conditions \(x_1(0) = 1, x_2(0) = 0, x_3(0) = 0\) is

Solution.

The matrix \(A\) is

\[
\begin{bmatrix}
-3 & 0 & -4 \\
-1 & -1 & -1 \\
1 & 0 & 1
\end{bmatrix}
\]

The characteristic equations is

\[
|A - \lambda I| = \begin{vmatrix}
-3 - \lambda & 0 & -4 \\
-1 & -1 - \lambda & -1 \\
1 & 0 & 1 - \lambda
\end{vmatrix} = -(\lambda + 1)^3
\]

Then the only eigenvalue we have is \(\lambda = -1\) of multiplicity 3. Thus we need a chain of generalized eigenvectors of length 3. We do it from top to bottom:

Set \((A - \lambda I)v = 0\). That is

\[
\begin{bmatrix}
-2 & 0 & -4 \\
-1 & 0 & -1 \\
1 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

This corresponds to three equations:

\[-2a - 4c = 0, -a - c = 0, a + 2c = 0\]

The first and the third equation are equivalent. We can only look at the first two. Solving this system we have that \(a = c = 0\) but we have no restriction to \(b\). To make \(v\) an eigenvector, we can set \(b = 1\) and so our eigenvector

\[
v_1 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

Now using \(v_1\) to find the next level \(v_2\) by solving the equation \((A - \lambda I)v_2 = v_1\):

\[
\begin{bmatrix}
-2 & 0 & -4 \\
-1 & 0 & -1 \\
1 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

Again this corresponds to equations:

\[-2a - 4c = 0, -a - c = 1, a + 2c = 0\]

Also the third equation is equivalent to the first one. Solving the first equations, we have that \(a = -2, c = 1\). We have no restriction to \(b\). So we may choose \(b = 0\). Thus this will give us a second-level generalized eigenvector

\[
v_2 = \begin{bmatrix}
-2 \\
0 \\
1
\end{bmatrix}
\]
Keep doing this to find \( v_3 \) by solving \((A - \lambda I)v_3 = v_2\):

\[
\begin{bmatrix}
-2 & 0 & -4 \\
-1 & 0 & -1 \\
1 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} = 
\begin{bmatrix}
-2 \\
0 \\
1
\end{bmatrix}
\]

This corresponds to equations:

\[-2a - 4c = -2, -a - c = 0, a + 2c = 1\]

Similarly we can choose \( a = -1, b = 0, c = 1 \) which give a solution to the above system. So the third generalized eigenvector is

\[ v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \]

Therefore we got a chain of generalized eigenvectors of length 3. Putting them into the formulas of independent solutions:

\[
X_1(t) = e^{\lambda t} v_1 = e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

\[
X_2(t) = e^{\lambda t}(v_1 t + v_2) = e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = e^{t} \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix}
\]

\[
X_3(t) = e^{\lambda t}(v_1 t^2/2 + v_2 t + v_3) = e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^2/2 + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = e^{t} \begin{bmatrix} -2t - 1 \\ t^2/2 \\ t + 1 \end{bmatrix}
\]

Thus by the fundamental theorem of differential system, the general solution to this system is the linear combination of all these independent solutions:

\[
X(t) = c_1 e^{-t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -2t - 1 \\ t^2/2 \\ t + 1 \end{bmatrix}
\]

or if we separate the rows:

\[
x_1(t) = -2c_2 e^{-t} + c_3 e^{-t} (-2t - 1)
\]

\[
x_2(t) = c_1 e^{-t} + c_2 e^{-t} t + c_3 e^{-t} t^2/2
\]

\[
x_3(t) = c_2 e^{-t} + c_3 e^{-t} (t + 1)
\]

Now since we have the initial condition \( X(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \), we plug in \( t = 0 \) to the above general solution to have a system of linear equations:

\[
x_1(0) = -2c_2 - c_3 = 1
\]

\[
x_2(0) = c_1 = 0
\]

\[
x_3(0) = c_2 + c_3 = 0
\]

It's easy to see that the solutions is \( c_1 = 0, c_2 = -1, c_3 = 1 \). Therefore, the particular to this IVP is

\[
X(t) = e^{-t} \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} -2t - 1 \\ t^2/2 \\ t + 1 \end{bmatrix} = e^{-t} \begin{bmatrix} -2t + 1 \\ t^2 - t \\ t \end{bmatrix}
\]
or equivalently

\[ x_1(t) = e^{-t}(-2t + 1) \]
\[ x_2(t) = e^{-t}\left(\frac{t^2}{2} - t\right) \]
\[ x_3(t) = e^{-t}t \]