1. Solve \( y' = \frac{2y}{1+x^2}, \ y(0) = 1. \)

**SOLUTION.** This equation is separable. We can separate the variables as follows:

\[
\frac{1}{2y} \, dy = \frac{1}{1-x^2} \, dx
\]

Integrate both sides with respect to their own variables to get

\[
\int \frac{1}{2y} \, dy = \int \frac{1}{1-x^2} \, dx
\]

This gives us the general solution

\[
\frac{1}{2} \ln y = \frac{1}{2} \ln(\frac{x+1}{x-1}) + C
\]

Simplify the solution:

\[
y = C \cdot \frac{x+1}{x-1}
\]

At the initial point \( y(0) = 1, \) we have \( 1 = C \cdot (-1). \) That is \( C = -1. \)

Thus the particular solution is given by

\[
y = \frac{1+x}{1-x}
\]

2. Solve \( xy' + 5y = 7x^2, \ y(2) = 5. \)

**SOLUTION 1.** Divide every term by \( x \) to normalize the equation:

\[
y' + \frac{5}{x}y = 7x
\]

Apparently this is a first order linear equation with \( P(x) = \frac{5}{x} \) and \( Q(x) = 7x. \)

We find the integrating factor

\[
\rho(x) = e^{\int \frac{5}{x} \, dx} = e^{5 \ln x} = x^5.
\]

Then the formula gives us that

\[
\rho(x) y(x) = \int \rho(x) Q(x) \, dx
\]

That is,

\[
x^2 y(x) = \int x^5 \cdot 7x \, dx = x^7 + C
\]
Thus the general solution is given by

\[ y = \frac{x^7 + C}{x^2} = x^5 + \frac{C}{x^2} \]

At the point \( y(2) = 5 \) we have \( 5 = 32 + \frac{C}{2^2} \). That is \( C = -108 \).
Therefore the particular solution is

\[ y(x) = x^5 - \frac{108}{x^2}. \]

3. Solve \( y' = \frac{4x^3 + 3y^2}{2xy} \), \( y(2) = 4 \).

**SOLUTION.** View this equation as a homogeneous equation:

\[ y' = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = \frac{2x}{y} + \frac{3y}{x} \]

Use substitution \( v = \frac{y}{x} \). The corresponding differential of \( y = vx \) is \( y' = v + xv' \). Thus the differential equation will become

\[ v + xv' = \frac{2}{v} + \frac{3v}{2} \]

That is \( xv' = \frac{v^2}{2} + \frac{2}{v} = \frac{v^2 + 4}{2v} \). This is a separable equation:

\[ \frac{2v}{v^2 + 4} dv = \frac{1}{x} dx \]

Thus

\[ \ln(v^2 + 4) = \ln x + C \]

and

\[ v^2 + 4 = Cx \]

Plug in our substitution back into the above solution:

\[ \frac{y^2}{x^2} + 4 = Cx \]

That is

\[ y^2 = Cx^3 - 4x^2 \]

Using the initial condition \( y(2) = 4 \), we have that \( 16 = C \cdot 8 - 4 \cdot 4 \), and hence \( C = 4 \). Therefore the particular solution is

\[ y = 4x^3 - 4x^2 \]

or equivalently

\[ y = \sqrt{4x^3 - 4x^2} \]

because the initial condition has positive \( y \)-value.

4. Solve the equation

\[ (2x + 3y) + (3x + 2y)y' = 0 \]
using the exact equation method.

**SOLUTION.** Check that this equation is exact by looking the appropriate partial derivatives.

Identify $M = 2x + 3y$ and $N = 3x + 2y$. Then $M_y = 3$ and $N_x = 3$. Then $M_y = N_x$ and hence this equation is exact.

Now the solution will be implicitly given by $F(x, y) = C$ for some function $F$ such that $F_x = M$ and $F_y = N$.

So integrate $M$ with respect to $x$ to get

$$F = x^2 + 3xy + h(y)$$

for some unknown function $h(y)$ of $y$. Then differentiate $F$ with respect to $y$. Since we know that $F_y = N$, this will give us an equation

$$3x + h'(y) = 3x + 2y.$$

Comparing terms on both sides, we have

$$h'(y) = 2y.$$

Since we only need one available function $F$, so we choose $h(y) = y^2$ by taking antiderivative. Thus $F(x, y) = x^2 + 3xy + y^2$ and the implicit general solution is given by

$$x^2 + 3xy + y^2 = C.$$

Using the initial condition $y(1) = 1$, i.e. when $x = 1$, we must have $y = 1$, we will get that $C = 1 + 1 + 1 = 5$.

Finally the particular implicit solution is

$$x^2 + 3xy + y^2 = 5.$$