MTH306T Project 4. Homogeneous Linear Systems

In this project, we are going to solve a homogeneous differential linear system \( x' = Ax \) in Maple.

New in Maple.
1. Load the package linalg in Maple by typing `with(linalg);`
2. Key Commands:
   - `matrix([ [Row 1], [Row 2], [Row 3] ]);` defines a 3*3 matrix.
   - `charpoly(A,x);` gives the characteristic polynomial of a matrix A.
   - `eigenvalues(A);` gives all the eigenvalues of a matrix A.
   - `eigenvectors(A);` gives the eigenvectors of a matrix A.
   - `inverse(A);` gives the inverse of a matrix A.
   - `evalm(A*B);` gives the product of matrices A and B.

Remarks.
1. **Use page 3 as the cover page of your project report.** Write your answers to the questions directly in the blanks and then **attach your Maple worksheet** to this report.
2. **Due date:** THU, Nov.21. You may hand in your project in recitation or my office hours: WED 1-2pm, THU 4:00-5:30pm at Math 136. No credit if you hand in late.

MTH306T Project 4 Instruction

1. Finding eigenvalues and eigenvectors of a matrix

Example. Suppose we have a matrix
\[
A = \begin{bmatrix}
3 & 2 & 2 \\
-5 & -4 & -2 \\
5 & 5 & 3
\end{bmatrix}
\]
and we want to find its characteristic polynomial, eigenvalues and eigenvectors.

With `with(linalg):`
The above command will load the linear algebra package in Maple so we can use some commands in that package.

\[
> A := \text{matrix}(\begin{bmatrix}
3 & 2 & 2 \\
-5 & -4 & -2 \\
5 & 5 & 3
\end{bmatrix});
\]

This command defines the matrix A in this example use the command `matrix()`. Every row is enclosed in brackets, and all three rows are enclosed in another pair of brackets.

\[
> \text{charpoly}(A, x);
\]
\[
(x - 3) \ (x^2 + x - 2)
\]
This command gives us the characteristic polynomial of matrix A. We use x as the variable in this polynomial.
These two commands give us the eigenvalues and eigenvectors of the matrix $A$. We can see there are three eigenvalues $3, -1, 2$ and corresponding eigenvectors are $[0, -1, 1]$, $[1, -1, 1]$, $[-1, 1, 0]$. 

2. Solving a linear system of algebraic equations.

Example. If we want to solve a system of linear equations to get $c_1, c_2, c_3$, such as

\[
\begin{align*}
3c_1 + 2c_2 + c_3 &= 6 \\
c_1 - c_2 - c_3 &= 0 \\
2c_1 + c_2 + 2c_3 &= 2
\end{align*}
\]

First we realize this system as a matrix system $M \cdot C = b$ where $M$ and $b$ are the two following matrices. We define these using \texttt{matrix( )}.

\[
M := \text{matrix}(\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & 2 \end{bmatrix});
\]

\[
b := \text{matrix}(\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix});
\]

Then by the theory of linear algebra, the solution $C$ can be obtained by multiply both sides by the inverse of $M$. Thus, $C = M^{-1} \cdot b$. We try to realize this process in Maple:

\[
N := \text{inverse}(M);
\]

\[
C := \text{evalm}(N \cdot b);
\]

This command will multiply the matrices $N$ and $b$, which will be our solution vector $C$. This means that $c_1 = 1$, $c_2 = 2$, and $c_3 = -1$. This is the solution we want for this linear system.
MTH306T project 4 Report

Part I: Solve for the general solution

Look at the linear system \( \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} \end{bmatrix} x \)

1. [Maple] Define this matrix in Maple.
2. [Maple] Find the characteristic equation, eigenvalues and corresponding eigenvectors of this matrix.
3. Then fill in the following blanks:

   The characteristic equation is:

   \[ \lambda_1 = ___, \quad v_1 = \begin{bmatrix} \_
\_
\_
\end{bmatrix}, \quad \lambda_2 = ___, \quad v_2 = \begin{bmatrix} \_
\_
\_
\end{bmatrix}, \quad \lambda_3 = ___, \quad v_3 = \begin{bmatrix} \_
\_
\_
\end{bmatrix} \]

   The general solution to this system is

   \[ x(t) = \ ]

Part II: Solve algebraic linear equations

Now we add the initial condition:

\[ x(0) = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} \]

1. Find the linear equations in terms of the unknown constants \( c_1, c_2, c_3 \):

   Your Answer:

   \[ \]

2. Transform this equation as a matrix equation of the form \( M \cdot C = b \).

   Your Answer:

   \[ \begin{bmatrix} \_
\_
\_
\_
\_
\end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \_
\_
\_
\end{bmatrix} \]

3. [Maple] Define this matrix \( M \) in Maple.

4. [Maple] Find the inverse matrix of \( M \). Write down the inverse:

   \[ \begin{bmatrix} \_
\_
\_
\_
\_
\end{bmatrix} \]

5. [Maple] Use this inverse matrix to find \( c_1, c_2, c_3 \). Write down your answer:

   \[ c_1 = ___, \quad c_2 = ___, \quad c_3 = ___, \]

6. Write down the final particular solution to this IVP:

   \[ x(t) = \ ]