MTH 538 Homework 4 (Spectral Methods)

Problem 1

Start with the formula for the inverse discrete Fourier transform and derive the coefficients of the spectral differentiation matrix, D, for a 2π periodic function, $u(x_j) = u_j$, defined on a uniform grid $x_j = jh$ with the step size $h = \frac{2\pi}{N}$ and $j = 1, \ldots, N$. Define the vector $\mathbf{u} = (u_1, \ldots, u_N)^T$ and $\mathbf{u}' = (u'_1, \ldots, u'_N)$, and the differentiation matrix satisfies,

$$\mathbf{u}' = D\mathbf{u}$$

You may leave the expression for the coefficients as a sum, or derive the expression for the coefficients in terms of elementary function for extra credit.

Problem 2

Derive the discrete orthogonality condition for complex exponentials,

$$\left(e^{ikx}, e^{ik'x}\right) = \begin{cases} 1, \ k = k'\\ 0, \ k \neq k' \end{cases}$$

where the discrete inner product is defined by,

$$(f,g) = \frac{1}{N} \sum_{j=1}^{N} f_j \bar{g}_j,$$

and bar denotes complex conjugation.

Problem 3

Consider the Strang splitting method, and derive the formula for the local truncation error for a system of ODEs,

$$\mathbf{y}' = A\mathbf{y} + B\mathbf{y}.$$

Assume that the matrices A and B do not commute.

Problem 4 Consider a spectral and a finite difference method for solving PDE,

$$iu_t = u_{xx} + |u|^2 u,$$

$$u(x, t = 0) = \cos x,$$

with periodic boundary conditions on $x \in [0, 2\pi]$. Implement the Strang splitting pseudo spectral method, and a suitable finite-difference

(FD) method of your choice. Make sure that the FD method that you choose is stable.

- Fix the size of the timestep τ that satisfies the stability criterion and compute the numerical solution at T = 1 by both methods. Compare the error obtained by both methods as a function of the spatial step $h = \frac{2\pi}{N}$. Report your findings in a graph of the error as a function of h.
- Repeat these steps, but now fix the spatial step h, and find how the error changes with the temporal step τ . Report your findings in a graph of the error as a function of τ .