Numerical Analysis Exam 1 Review Problems

1. Gauss Elimination

Solve the following system of linear equations using Gauss elimination:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 9 \\ 4x_1 + 7x_2 + 3x_3 &= 21 \\ 6x_1 + 9x_2 + 5x_3 &= 29 \end{aligned}$$

Find the values of x_1, x_2, and x_3 using back substitution.

2. LU Factorization

Factor the matrix A into LU:

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 8 & 3 \\ 6 & 12 & 5 \end{bmatrix}$$

After factorizing, solve the system $Ax = b$ where $b = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$.

3. Forward and Backward Substitution

Given the following matrices from LU factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

and $b = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$, solve for x using forward and backward substitution.

4. Algorithm Error Amplification

Given the matrix A and vector b for the system $Ax = b$:

$$A = \begin{bmatrix} 1.01 & 2 \\ 2.01 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Show how small changes in the input b (e.g., adding δb) can lead to large changes in the output x. Calculate the condition number of the matrix A.

5. Backward and Forward Relative Errors

Suppose we are solving a linear system where the true solution is $x_{true} = 2.1$, but due to computational errors, we get $x_{approx} = 2.08$. The data for the system is accurate to within 1%. Calculate the forward and backward relative errors for this system.

6. Decimal to Binary and Binary to Decimal Conversion

1. Convert the decimal number 35.125 into binary.
2. Convert the binary number 1011.011_2 into decimal.

7. Fraction to Binary Conversion

Convert the fraction $\frac{3}{7}$ into its binary representation.

8. Fixed Point Iteration

Given the equation $x = \cos(x)$, use fixed-point iteration starting with $x_0 = 0.5$. Perform 5 iterations and determine whether the method is converging.

9. Bisection Method

Use the bisection method to find the root of the function $f(x) = x^3 - x - 2$ in the interval $[1, 2]$. Perform 4 iterations and provide the approximate root after the iterations.

10. Newton's Method

Use Newton's method to find the root of the equation $f(x) = e^x - 2$ starting with $x_0 = 1$. Perform 3 iterations and provide the approximate root after each iteration.

11. Rootfinding (Simple and Multiple Roots)

1. Find the simple root of $f(x) = x^3 - 3x + 2$ using Newton's method starting at $x_0 = 2$.
2. Find the multiple root of $f(x) = (x - 1)^2$ using Newton's method starting at $x_0 = 2$. Compare the behavior of Newton's method for simple and multiple roots.