Practice Exam 2

Show all work and explain you calculations.

1. You need to calculate numerically a sum

$$\sum_{n=1}^{1000000} \frac{1}{2^n},$$

propose a method how to do this avoiding loss of significance (**Hint:** changing of order of summation could be beneficial). Explain you method.

2. What is machine epsilon ε_{mach} for IEEE 754 single precision standard (mantissa has 23 bits, 1 bit for sign, 8 bits for exponent)? Represent the answer in decimal notation.

3. Consider the function $f(x) = x^3(1 - \cos x)$.

(a) Find multiplicity for the root $r = 0$ of $f(x)$ and another root (any).

(b) Determine for both roots whether Newton's method or bisection will converge faster.

4. **(a)** Solve the system **finding LU-factorization and using two-back substitution**

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 &= 2, \\ x_1 + 5x_2 + 2x_3 &= 1, \\ 4x_1 - x_2 + 9x_3 &= 12. \end{cases}$$

(b) Estimate how many floating point operations have you used. How this number will change for 6×6 system?

5. **(a)** Given the data:

x	0	3	6
y	1	4	5

find the best least squares fit by a linear function **using Givens' rotations for QR factorization (!)**.

(b) What is a residue (error) of least squares approximation?

6. Reminder: $\cos(\pi/6) = \sqrt{3}/2,\ \cos(\pi/3) = 1/2$.

(a) Interpolate function $f(x) = \cos x$ with a polynomial on an interval $[0, \pi/3]$ using equally space points $x = \{0, \pi/6, 2\pi/6\}$ using Newton's divided differences. Bring your answer to the $a_n x^n + a_{n-1}x^{n-1} + \ldots$ form.

(b) Do the same using Lagrange approach.

7. **(a)** Integrate function $f(x) = x^3$ on the interval $[0, 1]$ analytically **AND** numerically, using Simpson method (think how many points you need). Explain the result.

(b) Calculate second derivative of the same function at the point $x = 1/2$ analytically and using three-point centered-difference formula with grid step $h = 1/2$. Explain the result.

8. **(a)** Derive a nonlinear system of equations for points positions and weights of three point Gauss-Legendre integration scheme for integration on the interval $[0, 1]$.

(b) Estimate error of numerical integration using the scheme you just derived for function $f(x) = x^5 - 3x^3$.

9. **(a)** Write down any method for numerical solution of an initial value problem for the first order ODE

$$y' = f(t, y), \quad y(0) = y_0.$$

with local truncation error of the order of $O(h^2)$.

(b) Write down any method for numerical solution of an initial value problem for the same first order ODE with local truncation error of the order of $O(h^3)$.

(c) Estimate how global truncation error of the previously described numerical methods for solution of ODEs will change is we consider three times finer time step (for simplicity consider homogeneous time steps).