1. Sauer, Exercise 4a, page 85 (by hand).

2. Sauer, Exercise 7, page 85 (use the estimate $2n^3/3$ for full Gaussian elimination obtained in class and similar estimate for upper-triangular matrix).

3. Write the following MATLAB® functions.

1. `function [L,U]=naivege(A)` that returns the lower triangular matrix L and an upper triangular matrix U such that $A = LU$, obtained by Gauss elimination without pivoting.

2. `function x=utrisol(U,b)` that solves the upper triangular system $U\mathbf{x} = \mathbf{b}$ using backward substitution.

3. `function x=ltrisol(L,b)` that solves the lower triangular system $L\mathbf{x} = \mathbf{b}$ using forward substitution.

Use the above MATLAB® functions to solve the system from **1.** Remember, to use the LU decomposition to solve $A\mathbf{x} = \mathbf{b}$ you take the two steps:

1. Solve $L\mathbf{y} = \mathbf{b}$

2. Solve $U\mathbf{x} = \mathbf{y}$

4. This problem explores the relationship between condition number and the volume of a random parallelepiped in 4-dimensional space whose sides are unit vectors. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ be vectors which emanate from a given vertex of the parallelepiped and describe its sides. By a well-known result from geometry, the volume of the parallelepiped is then $\left|\det([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4])\right|$. Your program for this problem should do the following.

(a) Generate 4 random column vectors of size 4-by-1, viewing them as the 4 columns of a 4-by-4 random matrix `A = rand(4)`.

(b) Normalize the columns `A = A*diag(1./sqrt(sum(A.*A)))`. Provide some comments to explain what this operation does.

(c) Compute the volume of the parallelepiped defined by the normalized vectors. Your program must use the LU factorization for computing the determinant (that is do not use the command `det(A)` here).

(d) Compute the condition number $\kappa(A)$ with `cond(A)`.

(e) Repeat the process 1000 times, and produce a scatter plot of $\det(A)$ versus $1/\kappa(A)$.

What are the maximum and minimum values found for $|\det(A)|$? What is the condition number corresponding to each of these values? How can you explain the extreme values of the condition number and the corresponding values of the determinant?

1. Exercise 6(a,c,e), page 93, Sauer's textbook (by hand).

2. Given a vector $\mathbf{x} \in \mathbb{R}^n$ with distinct elements (that is, $x_j \neq x_k$ for $j \neq k$), the *Cauchy matrix* $C(x)$ is the n-by-n matrix with entries

$$c_{ij} = \frac{1}{x_i + x_j}.$$

Write a MATLAB® function which returns the Cauchy matrix for a random input vector `x = rand(n,1)`. This can be achieved in one line with `repmat`. For $n = 4, 8, 12, 16$, do the following. Compute the corresponding $C(x)$. Define $\mathbf{z}_{\mathrm{exact}} = [1, 1, \ldots, 1]^T =$ `ones(n,1)`, and then compute $\mathbf{b} = C\mathbf{z}_{\mathrm{exact}}$. Using MATLAB's backslash, *numerically* solve the equation

$$C\mathbf{z} = \mathbf{b},$$

thereby producing a *computed* solution $\mathbf{z}_{\mathrm{c}}$. In exact arithmetic $\mathbf{z}_{\mathrm{c}} = \mathbf{z}_{\mathrm{exact}}$ of course, but in IEEE double precision $\mathbf{z}_{\mathrm{c}}$ will not equal $\mathbf{z}_{\mathrm{exact}}$. The quantity $\|\mathbf{z}_{\mathrm{c}} - \mathbf{z}_{\mathrm{exact}}\|_\infty$ is the forward error. Construct a table which, for each n, collects the relative forward error, relative backward error, magnification factor, and (infinity-norm) condition number of C. Discuss your results.