MTH 637 — Homework 2

Topic: Distributions and Fourier Transform

Use the Fourier transform convention

$$\widehat{f}(k) = \int_{\mathbb{R}} f(x)e^{-ikx}\,dx, \qquad f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(k)e^{ikx}\,dk.$$

Problem 1 — Fourier transform of basic distributions

1. Compute $\widehat{\delta}$ and $\widehat{\delta'}$.
2. Show that
$$\widehat{1} = 2\pi\delta.$$
3. Let $H(x)$ be the Heaviside function. Using $H' = \delta$, compute $\widehat{H}$ in the sense of distributions.
4. Compute the Fourier transform of $\operatorname{sgn}(x)$.

Problem 2 — Distributional derivatives

1. Show that
$$H' = \delta.$$
2. Compute the distributional derivative of $|x|$.
3. Compute the second distributional derivative of $|x|$.

Problem 3 — Fourier transform identities

Prove:

1. $$\widehat{f'}(k) = ik\widehat{f}(k),$$
2. $$\widehat{xf}(k) = i\frac{d}{dk}\widehat{f}(k),$$
3. $$\widehat{f''}(k) = -k^2\widehat{f}(k).$$

Problem 4 — Fundamental solution via Fourier transform

Solve in the sense of distributions:

$$-u'' + u = \delta.$$

1. Take the Fourier transform and solve for $\widehat{u}(k)$.
2. Invert the transform to obtain $u(x)$.
3. Verify directly that your solution satisfies the equation.

Problem 5 — Heat equation with singular initial data

Consider

$$u_t - u_{xx} = 0, \qquad u(x,0) = \delta(x).$$

1. Solve using Fourier transform in x.
2. Show that
$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$
3. Show that $u(\cdot, t) \to \delta$ in the sense of distributions as $t \to 0^+$.