Homework 1

Due Feb 24, 2026

Instructions. Show all steps and justify your arguments. You may assume sufficient smoothness and decay of functions whenever needed to justify differentiations under the integral sign or interchange of limits.

Problem 1: Method of Characteristics with Discontinuous Velocity

Consider the Cauchy problem

$$u_t + a(x)\, u_x = 0, \qquad (x,t) \in \mathbb{R} \times (0,\infty),$$

with initial condition

$$u(x,0) = u_0(x),$$

where

$$a(x) = \begin{cases} 1, & x < 0, \\ 2, & x > 0, \end{cases} \qquad u_0(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

1. Solve the equation using the method of characteristics.
2. Derive an explicit formula for $u(x,t)$.
3. Sketch the solution $u(\cdot,t)$ for a fixed $t > 0$.
4. Identify the curves in the (x,t)–plane across which the solution is discontinuous and explain their origin.

Problem 2: Nonlinear Transport and Breakdown of Classical Solutions

Consider the inviscid Burgers equation

$$u_t + u\, u_x = 0, \qquad u(x,0) = \phi(x),$$

where $\phi \in C^1(\mathbb{R})$.

1. Use the method of characteristics to show that the solution is given implicitly by

$$u(x,t) = \phi(\xi), \qquad x = \xi + t\,\phi(\xi).$$

2. Show that the solution remains classical as long as

$$1 + t\,\phi'(\xi) \neq 0 \quad \text{for all } \xi.$$

3. Assuming $\min_\xi \phi'(\xi) < 0$, show that the first time of gradient blow-up is

$$t_* = -\frac{1}{\min_\xi \phi'(\xi)}.$$

4. Compute t_* explicitly for $\phi(x) = \sin x$, and determine where the first singularity forms.

Problem 3: Laplace Equation on a Rectangle

Solve Laplace's equation on the rectangle $(0, L) \times (0, H)$:

$$u_{xx} + u_{yy} = 0,$$

subject to boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = f(x),$$

where $f \in L^2(0, L)$.

1. Use separation of variables to derive a series solution for $u(x, y)$.
2. Show that the solution can be written as

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big) \frac{\sinh\big(\frac{n\pi y}{L}\big)}{\sinh\big(\frac{n\pi H}{L}\big)}.$$

3. Express the coefficients b_n in terms of the boundary data f.
4. Compute b_n explicitly when $f(x) = x(L - x)$.

Problem 4: Fundamental Solution of the Poisson Equation in $\mathbb{R}^3$

Derive the fundamental solution of the Poisson equation in three dimensions:

$$-\Delta\Phi = \delta_0 \quad \text{in } \mathbb{R}^3.$$

1. Assume Φ is radially symmetric and solve $\Delta\Phi = 0$ for $|x| > 0$.
2. Use the divergence theorem on a ball B_ε to determine the constant by matching the total flux through ∂B_ε.
3. Show that

$$\Phi(x) = \frac{1}{4\pi|x|}.$$