Take-Home Midterm 3

Due May 11

Problem 1 — FPU chain: dispersion and KdV limit

Consider the FPU-α chain

$$\ddot{q}_n = q_{n+1} - 2q_n + q_{n-1} + \alpha \left[(q_{n+1} - q_n)^2 - (q_n - q_{n-1})^2 \right],$$

for $n = 1, \ldots, N$, with fixed ends

$$q_0 = q_{N+1} = 0.$$

(a) Set $\alpha = 0$. Show that the normal modes are

$$q_n(t) = A_k(t) \sin\left(\frac{\pi k n}{N+1}\right),$$

derive the dispersion relation ω_k, compute the group velocity $c_g(k)$, and determine the behavior of ω_k for small k.

(b) Let

$$\xi = \frac{\pi k}{N+1}.$$

Rewrite the dispersion relation in terms of ξ. Compare it with the dispersion relation of the wave equation

$$u_{tt} = u_{xx}.$$

Identify the limit in which the discrete system recovers dispersion of the wave equation.

(c) Now take $\alpha \neq 0$. Introduce

$$r_n = q_{n+1} - q_n.$$

Derive

$$\ddot{r}_n = \Delta_d(r_n + \alpha r_n^2), \qquad \Delta_d r_n = r_{n+1} - 2r_n + r_{n-1}.$$

Then use

$$r_n(t) = \varepsilon^2 u(X, T), \qquad X = \varepsilon(n - t), \qquad T = \varepsilon^3 t,$$

to derive a KdV-type equation for $u(X, T)$.

Problem 2 — Virial identity

Consider the focusing nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{2p}u = 0, \quad x \in \mathbb{R}^d.$$

Define

$$I(t) = \int_{\mathbb{R}^d} |x|^2 |u(x,t)|^2 \, dx.$$

(a) Show that

$$\frac{d}{dt} I(t) = 4 \operatorname{Im} \int x \cdot \nabla u \, \overline{u} \, dx.$$

(b) Show that

$$I''(t) = 8\|\nabla u\|_2^2 - \frac{4dp}{p+1}\|u\|_{2p+2}^{2p+2}.$$

Rewrite your result in terms of the energy

$$E(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2p+2}\|u\|_{2p+2}^{2p+2}.$$

(c) Explain what happens in the case $dp = 2$. How does this relate to stability or instability of solutions?

Problem 3 — Scaling and stability

Consider the focusing NLS:

$$iu_t + \Delta u + |u|^{2p}u = 0.$$

(a) Using the mass-preserving scaling

$$u_\lambda(x) = \lambda^{d/2} u(\lambda x),$$

determine how the kinetic and potential energy scale with λ.

(b) Classify the equation as subcritical, critical, or supercritical in terms of d and p.

(c) Consider the energy along the scaling orbit

$$E(\lambda) = E(u_\lambda).$$

Determine whether $E(\lambda)$ has a minimum, maximum, or no extremum in each regime.

(d) Explain how the behavior of $E(\lambda)$ relates to the stability or instability of soliton solutions.

(e) Explain why the scaling argument alone is inconclusive in the critical case $dp = 2$. Why is the virial identity needed to explain stability in the critical case?

Problem 4 — FPU Hamiltonian in normal modes

Consider the FPU-α Hamiltonian

$$H = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \sum_{n=0}^{N}\left[\frac{1}{2}(q_{n+1} - q_n)^2 + \frac{\alpha}{3}(q_{n+1} - q_n)^3\right],$$

with fixed ends

$$q_0 = q_{N+1} = 0.$$

Define

$$q_n = \sqrt{\frac{2}{N+1}}\sum_{k=1}^{N} Q_k \sin\left(\frac{\pi k n}{N+1}\right),$$

$$p_n = \sqrt{\frac{2}{N+1}}\sum_{k=1}^{N} P_k \sin\left(\frac{\pi k n}{N+1}\right).$$

(a) Show that Hamilton's equations in the new variables take the canonical form

$$\dot{Q}_k = \frac{\partial H}{\partial P_k}, \qquad \dot{P}_k = -\frac{\partial H}{\partial Q_k}.$$

(b) Rewrite the quadratic part of the Hamiltonian as

$$H_2 = \frac{1}{2}\sum_{k=1}^{N}\left(P_k^2 + \omega_k^2 Q_k^2\right),$$

and determine ω_k.

(c) Let

$$r_n = q_{n+1} - q_n.$$

Express r_n in terms of Q_k, and show that

$$H_3 = \frac{\alpha}{3}\sum_{i,j,k=1}^{N} C_{ijk} Q_i Q_j Q_k.$$

Give an explicit expression for C_{ijk}.

(d) Use Hamilton's equations to derive

$$\ddot{Q}_k + \omega_k^2 Q_k = -\alpha \sum_{i,j=1}^{N} C_{kij} Q_i Q_j.$$

Explain briefly how this equation shows that the nonlinearity couples the normal modes.