Take-Home Exam 2

Problem 1

Let $H(x)$ be the Heaviside function.

1. Show that $H'(x) = \delta(x)$ in the sense of distributions.
2. Compute the Fourier transform of $H(x)$.
3. Show
$$\hat{H}(k) = \pi\delta(k) + \text{p.v.}\left(\frac{1}{ik}\right).$$

Problem 2

Consider
$$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad u(x,0) = f(x).$$

1. Take Fourier transform in x and solve for $\hat{u}(k,t)$.
2. Invert the transform and show
$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy.$$
3. In what sense does $u(x,t) \to f(x)$ as $t \to 0^+$?

Problem 3

Solve
$$u_t = u_{xx} + \delta(x), \quad u(x,0) = 0.$$

1. Solve in Fourier space.
2. Invert to obtain an integral formula for $u(x,t)$.
3. Briefly interpret the solution.

Problem 4

Consider

$$u_{tt} - u_{xx} = 0, \quad u(x,0) = 0, \quad u_t(x,0) = g(x).$$

1. Take Laplace transform in time.
2. Solve the resulting equation in Fourier space.
3. Invert to recover the solution.

Problem 5

Solve

$$-u''(x) = \delta(x).$$

1. Solve using Fourier transform.
2. Show that

$$u(x) = -\frac{1}{2}|x|.$$

3. In what sense does this solve the equation?

Problem 6

Consider p.v.$\left(\frac{1}{x}\right)$.

1. Compute its Fourier transform and show

$$\widehat{\text{p.v.}(1/x)} = -i\pi \,\text{sgn}(k).$$

2. Explain the relation to the Fourier transform of $H(x)$.