# PDE Take-Home Exam

Solve the following problems. Clearly justify all steps in your derivations.

**Problem 1. Bound States of the Hydrogen Atom**

Consider the time-independent Schrödinger equation

$$-\frac{1}{2}\Delta\psi - \frac{1}{r}\psi = E\psi$$

in $\mathbb{R}^3$.

- Rewrite the equation in spherical coordinates.
- Use separation of variables

  $$\psi(r,\theta,\phi) = R(r)Y_\ell^m(\theta,\phi)$$

  and derive the radial equation for $R(r)$.
- Show that bounded solutions exist only for discrete negative energies. Only consider angle independent states ($\ell = 0$).
- Determine the formula for the allowed energy levels, $E$, and write the associated wave-function $\psi(r,t)$

**Problem 2. Quantum Harmonic Oscillator**

Consider the equation

$$-\psi'' + \omega^2 x^2 \psi = E\psi, \qquad x \in \mathbb{R}.$$

- Show that solutions may be written in the form

  $$\psi(x) = e^{-x^2/2}H(x)$$

  for a suitable function $H(x)$.
- Derive the differential equation satisfied by $H(x)$.
- Show that bounded solutions occur only when $H$ is a polynomial.
- Determine the corresponding energy spectrum.

**Problem 3. Fundamental Solution of the Heat Equation**

Consider the heat equation on the real line

$$u_t = \kappa u_{xx}, \qquad x \in \mathbb{R},\ t > 0,$$

with initial condition

$$u(x,0) = f(x).$$

- Apply the Fourier transform in $x$

  $$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx$$

  and show that $\hat{u}(k,t)$ satisfies

  $$\frac{d}{dt}\hat{u}(k,t) = -\kappa k^2 \hat{u}(k,t).$$

- Solve this ODE and express $\hat{u}(k,t)$ in terms of $\hat{f}(k)$.
- Compute the inverse Fourier transform to show that the solution can be written as

  $$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)f(y)\,dy$$

  for some function $K(x,t)$.
- Show that

  $$K(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right).$$

- Verify directly that $K(x,t)$ satisfies the heat equation.