MTH 637: Advanced Numerical Analysis I
Homework 2: Numerical Methods for PDEs

Instructions. This assignment studies numerical methods for model partial differential equations. You are expected to combine analysis (consistency, stability, accuracy) with computational experiments. Clearly justify all steps. When numerical results are requested, include plots and brief discussion.

You may use any programming language.

Problem 1: Linear Advection

Consider the linear advection equation

$$u_t + a u_x = 0, \qquad x \in [0, 2\pi], \quad t \geq 0,$$

with periodic boundary conditions and initial data

$$u(x, 0) = \sin x + \tfrac{1}{2} \sin(3x).$$

(a) Derive the following numerical schemes:

- Upwind scheme (assume $a > 0$),
- Lax–Friedrichs scheme,
- Lax–Wendroff scheme.

(b) Determine the order of accuracy of each scheme.

(c) Perform a Von Neumann stability analysis for each method and state the corresponding stability condition.

(d) Implement all three schemes and compute the solution up to a fixed time $t = T$.

(e) Compare the numerical solutions with the exact solution. Comment on:

- numerical diffusion,
- phase (dispersion) error,
- overall accuracy.

(f) Perform a convergence study by refining Δx and Δt appropriately. Verify the observed order of convergence.

Problem 2: Heat Equation

Consider the heat equation

$$u_t = \kappa u_{xx}, \qquad x \in [0,1], \quad t \geq 0,$$

with boundary conditions $u(0,t) = u(1,t) = 0$ and initial condition

$$u(x,0) = \sin(\pi x).$$

(a) Derive the following fully discrete schemes:

- Forward Euler in time with centered differences in space,
- Backward Euler,
- Crank–Nicolson.

(b) Determine the local truncation error of each scheme.

(c) Analyze the stability of each method. In particular:

- derive the time step restriction for the explicit method,
- explain why the implicit methods are stable.

(d) Implement all three methods and compute the solution up to time $t = T$.

(e) Compare with the exact solution

$$u(x,t) = e^{-\kappa \pi^2 t} \sin(\pi x).$$

(f) Discuss the qualitative differences between the methods, in particular:

- numerical damping,
- behavior for large time steps.

Problem 3: Modified Equation

Consider a finite difference scheme for the advection equation from Problem 1.

(a) Derive the modified equation up to second order in Δx and Δt for one of the following schemes:

- Upwind scheme, or
- Lax–Friedrichs scheme.

(b) Identify the leading-order error term and interpret it as an artificial diffusion or dispersion term.

(c) Based on the modified equation, predict how the numerical solution should behave for non-smooth initial data.

(d) Verify your prediction numerically using a discontinuous initial condition (for example, a square wave). Briefly compare the observed behavior with your analysis.

Problem 4: Poisson Equation in Two Dimensions

Consider the Poisson equation

$$-\Delta u = f(x, y), \qquad (x, y) \in (0, 1)^2,$$

with Dirichlet boundary conditions chosen so that the exact solution is

$$u(x, y) = \sin(\pi x) \sin(\pi y).$$

(a) Determine the corresponding right-hand side $f(x, y)$.

(b) Derive the standard five-point finite difference discretization.

(c) Write the resulting linear system in matrix form. Describe its structure.

(d) Solve the system numerically on a sequence of grids.

(e) Verify second-order convergence in space.