Numerical PDEs – Exam Review Problems

These problems review topics from upwind/downwind schemes through exponential time differencing.

Problem 1: Upwind vs Downwind

Consider

$$u_t + u_x = 0$$

on a periodic grid.

1. Write the upwind scheme.
2. Write the downwind scheme.
3. Perform Von Neumann stability analysis for both schemes.
4. Determine the CFL condition.
5. Which scheme is unstable? Explain why.

Problem 2: Lax–Friedrichs Modified Equation

Consider

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n).$$

1. Expand using Taylor series.
2. Derive the modified equation.
3. Identify the artificial diffusion coefficient.
4. How does diffusion depend on Δx and Δt?

Problem 3: Lax–Wendroff Dispersion

Consider the Lax–Wendroff scheme.

1. Perform Von Neumann stability analysis.
2. Compute the amplification factor.
3. Expand for small wavenumbers.

4. Does the scheme introduce diffusion or dispersion?

5. Compare with Lax–Friedrichs.

Problem 4: Discrete Fourier Transform

Consider periodic grid

$$x_j = j\Delta x, \quad j = 0, \ldots, N-1.$$

1. Define the discrete Fourier transform.

2. Define the inverse transform.

3. Derive the spectral differentiation matrix.

4. Show the spectral differentiation rule

$$\partial_x u \leftrightarrow ik\hat{u}.$$

Problem 5: Aliasing

Consider $N = 4$ grid points.

1. Evaluate e^{i3x_j} on the grid.

2. Show it equals a lower-frequency mode.

3. Which wavenumbers cannot be distinguished?

4. What is the highest resolvable wavenumber?

Problem 6: Spectral vs Finite Difference Eigenvalues

Consider differentiation matrices on a periodic grid and a central-difference approximation to derivative,

$$(Du)_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x}.$$

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1. Apply spectral differentiation matrix to Fourier mode and verify that each discretized Fourier mode is an eigenvector.

2. Apply central-difference differentiation matrix to Fourier mode and verify that each discretized Fourier mode is an eigenvector.

3. Find eigenvalues of both matrices and compare.

4. Which one offers a better approximation to the derivative operator and why?

5. What happens near the smallest and the largest wavenumbers?

Problem 7: Integrating Factor Method

Consider

$$u_t = Lu + N(u).$$

1. Derive the integral equation satisfied by the solution,

$$u_{n+1} = e^{Lh}u_n + \int_0^h e^{L(h-\tau)} N(u(t_n+\tau))d\tau.$$

Problem 8: Split-Step Method

Consider

$$u_t = Lu + N(u),$$

where L is the linear part and $N(u)$ is the nonlinear part. Assume that the subproblems

$$u_t = Lu \qquad \text{and} \qquad u_t = N(u)$$

can each be solved exactly.

1. Write a first-order split-step method and show why this method is first-order accurate.
2. Write the Strang splitting method and show why Strang splitting is second-order accurate

$$u^{n+1} = e^{\frac{h}{2}L}\,\Phi_h^N\,e^{\frac{h}{2}L}u^n.$$

   where $u(s) = \Phi_s^N u(t)$ solves $u_t = N(u)$ on $s \in [t, t+h]$

Problem 9: Exponential Time Differencing

Using the result above:

1. Approximate nonlinear term as constant to derive ETD1 scheme.

$$u_{n+1} = e^{Lh}u_n + h\varphi_1(Lh)N(u_n).$$

2. Show

$$\varphi_1(z) = \frac{e^z - 1}{z}.$$

3. Derive ETD2 scheme by approximating the nonlinear term linearly,

$$N(u(t_n+s)) = N(u(t_n)) + s(N(u(t_{n+1})) - N(u(t_n)))$$

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