MTH 538 — Review Problems

Problem 1 — Second-Order Taylor Method

Consider the IVP

$$y' = e^t - y^2, \qquad y(0) = 1.$$

1. Derive the second-order Taylor method for this equation.
2. Write the update formula explicitly.
3. Determine the order of the local truncation error.

Problem 2 — Stability Near Equilibria

Consider

$$y' = \lambda y - y^3, \qquad \lambda > 0.$$

1. Find all equilibrium solutions.
2. Linearize the equation near each equilibrium.
3. Determine which equilibria are stable.
4. Apply the Forward Euler method near a stable equilibrium and determine for which values of h the numerical solution converges.
5. Repeat for the Backward Euler method.

Problem 3 — Von Neumann Stability

Consider the heat equation

$$u_t = \nu u_{xx}.$$

Discretize using forward Euler in time and centered differences in space:

$$u_j^{n+1} = u_j^n + r\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right), \quad r = \frac{\nu \Delta t}{\Delta x^2}.$$

1. Perform a Von Neumann stability analysis.
2. Compute the amplification factor.
3. Determine the restriction on r required for stability.

Problem 4 — Second-Order Runge–Kutta Method

Consider the Runge–Kutta method with Butcher tableau

$$\begin{array}{c|cc} 0 & & \\ \alpha & \alpha & \\ \hline & 1-\frac{1}{2\alpha} & \frac{1}{2\alpha} \end{array}$$

1. Write the method explicitly.
2. For what values of α is the method second order?
3. Identify the value of α corresponding to the midpoint method.

Problem 5 — Linear Multistep Method

Consider the two-step method

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1}).$$

1. Show that the method is consistent.
2. Determine its order of accuracy.
3. Determine whether the method converges.
4. Determine the region of absolute stability for the test equation $y' = \lambda y$.

Problem 6 — Crank–Nicolson for the Heat Equation

Consider the heat equation

$$u_t = \nu u_{xx},$$

and the Crank–Nicolson discretization

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\nu}{2}\left(\delta_{xx} u_j^{n+1} + \delta_{xx} u_j^n\right),$$

where

$$\delta_{xx} u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}.$$

1. Show that the method is second order in time.
2. Show that it is second order in space.
3. Conclude the overall order of accuracy of the full discretization.