Problems on Numerical Integration Methods

Problem 1: Applying Newton-Cotes Formulas

Use the **trapezoidal rule** (a Newton-Cotes method) to approximate the integral:

$$\int_0^2 x^3 \, dx$$

Divide the interval $[0, 2]$ into $n = 4$ subintervals. Compare the result with the exact value of the integral.

Problem 2: Gauss-Legendre Quadrature

Use the **2-point Gauss-Legendre quadrature rule** to approximate the integral:

$$\int_{-1}^1 \frac{1}{1+x^2} \, dx$$

Compare the result with the exact value of the integral, which is $\arctan(1) - \arctan(-1) = \pi/2$.

Problem 3: Comparing Accuracy

Consider the integral:

$$\int_0^1 e^{-x^2} \, dx$$

1. Approximate the integral using the **Simpson's rule** (a Newton-Cotes method) with $n = 4$ subintervals.

2. Approximate the integral using **3-point Gauss-Legendre quadrature**.

3. Discuss which method is more accurate and why, based on the nature of the function e^{-x^2}.

Problem 4: Deriving Newton-Cotes Weights for Simpson's Rule

Derive the weights for the **Simpson's rule** (Newton-Cotes formula for $n = 3$) based on the condition that the formula integrates polynomials up to degree 2 exactly.