Numerical Analysis Exam 2 Review Problems

1a Given points $(1,2),(2,3),(3,5)$, construct the interpolating polynomial in monomial form and write it as $p(x) = a_0 + a_1x + a_2x^2$.

1b Describe the error formula for monomial interpolation and explain how it depends on the degree of the polynomial and the spacing of interpolation points.

2a Using the points $(0,1),(1,2),(2,4)$, construct the interpolating polynomial using the Newton divided differences method.

2b Given an interpolating polynomial $p(x)$ built from points $(0,1),(1,2),(2,4)$, explain how you would extend the polynomial if an additional point $(3,8)$ is added without recalculating the entire polynomial from scratch.

3a Given points $(1,2),(3,6),(5,10)$, construct the Lagrange interpolating polynomial.

3b For the points $(1,1),(2,4),(3,9)$, construct the Lagrange polynomial and evaluate it at $x = 2.5$. Explain any tricks to simplify the computation.

4 Construct the Chebyshev interpolating polynomial for the function $f(x) = \frac{1}{1+25x^2}$ on the interval $[-1,1]$ using the Chebyshev nodes for $n = 4$. Write out the polynomial explicitly and discuss the advantages of using Chebyshev nodes over equally spaced nodes in minimizing interpolation error.

5 Given data points $(1,2),(2,3),(3,5),(4,4)$, construct the natural cubic spline that interpolates these points. Write out the system of equations needed to solve for the spline coefficients, and describe the conditions that make the spline "natural."

6 Given the matrix

$$A = \begin{pmatrix} 6 & 5 \\ 5 & -2 \\ 0 & 3 \end{pmatrix},$$

use Givens rotations to reduce A to an upper triangular matrix R, and find the orthogonal matrix Q such that $A = QR$.

7 Given the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix},$$

use Gram-Schmidt orthogonalization to find the matrices Q and R in the QR factorization $A = QR$.

8 Given the overdetermined system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},$$

use QR factorization to find the least squares solution for x and find the norm of the residual.