Extension field

*F* is a field.

A field *E* for which  and for which the operations of *F* are those of *E* restricted to *F*.

Fundamental Theorem of Field Theory

(Kronecker’s Theorem)

Let *F* be a field and  a nonconstant polynomial in . Then there is an extension field *E* of *F* in which  has a zero.

 splits in *E*

*E* is an extension field of *F*.

 can be factored as a product of linear factors in .

Splitting field for  over *F*

*F* is a field.

An extension field *E* of *F* in which  splits, but for which  does not split in any proper subfield of *E*.

Existence of Splitting Fields

Let *F* be a field and let  be a nonconstant element of . Then there exists a splitting field *E* for  over *F*.



Let *F* be a field and  be irreducible over *F*. If *a* is a zero of  in some extension *E* of *F*, then  is isomorphic to . Furthermore, if , then every member of  can be uniquely expressed in the form



where .



Let *F* be a field and  be irreducible over *F*. If *a* is a zero of p(x) in some extension *E* of *F* and *b* is a zero of p(x) in some extension *E’* of *F*, then the fields  and  are isomorphic.

Lemma, p. 351

Let *F* be a field, let  be irreducible over *F*, and let *a* be a zero of  in some extension of *F*. If  is a field isomorphism from *F* to *F’* and *b* is a zero of  in some extension of *F’*, then there is an isomorphism from  to  that agrees with  on *F* and carries *a* to *b*.

Extending 

Let  be an isomorphism from a field *F* to a field *F’* and let . If *E* is a splitting field for  over *F* and *E’* is a splitting field for  over *F’*, then there is an isomorphism from *E* to *E’* that agrees with  on *F*.

Splitting Fields Are Unique

Let *F* be a field and let . Then any two splitting fields of  over *F* are isomorphic.

Derivative

Let  belong to .

The polynomial  in .

Properties of the Derivative

Let  and let . Then

1. .
2. .
3. 

Criterion for Multiple Zeros

A polynomial  over a field *F* has a multiple zero in some extension *E* if and only if  and  have a common factor of positive degree in .

Zeros of an Irreducible

Let  be an irreducible polynomial over a field *F*. If *F* has characteristic 0, then  has no multiple zeros. If *F* has characteristic , then  has a multiple zero only if it is of the form  for some .

Perfect field

A field *F* with characteristic 0 or with characteristic *p* and .

Finite Fields Are Perfect

Every finite field is perfect.

Criterion for No Multiple Zeros

If  is an irreducible polynomial over a perfect field *F*, then  has no multiple zeros.

Zeros of an Irreducible over a Splitting Field

Let  be an irreducible polynomial over a field *F* and let *E* be a splitting field of  over *F*. Then all the zeros of  in *E* have the same multiplicity.

Factorization of an Irreducible over a Splitting Field

Let  be an irreducible polynomial over a field *F* and let *E* be a splitting field of . Then  has the form , where  are distinct elements of *E* and .