Ring homomorphism

A mapping from a ring *R* to a ring *S* that preserves the two ring operations; that is, for all , we have  and .

Ring isomorphism

A ring homomorphism that is both one-to-one and onto.

Properties of ring homomorphisms

Let  be a ring homomorphism from a ring *R* to a ring *S*. Let *A* be a subring of *R* and *B* an ideal of *S*.

1. For any  and any positive integer *n*,   and .
2.  is a subring of *S*.
3. If *A* is an ideal and  is onto *S*, then  is an ideal.
4.  is an ideal of *R*.
5. If *R* is commutative, then  is commutative.

More properties of ring homomorphisms

Let  be a ring homomorphism from a ring *R* to a ring *S*. Let *A* be a subring of *R* and *B* an ideal of *S*.

If *R* has a unity 1, , and  is onto, then  is the unity of *S*.

1.  is an isomorphism if and only if  is onto and .
2. If  is an isomorphism from *R* onto *S*, then  is an isomorphism from *S* onto *R*.

Kernels are Ideals

Let  be a homomorphism from a ring *R* to a ring *S*. Then  is an ideal of *R*.

First Isomorphism Theorem for Rings

a.k.a.

Fundamental Theorem of Ring Homomorphisms

Let  be a ring homomorphism from *R* to *S*. Then the mapping from  to , given by , is an isomorphism. In symbols, .

Ideals are Kernels

Every ideal of a ring *R* is the kernel of a ring homomorphism of *R*. In particular, an ideal *A* is the kernel of the mapping  from *R*  to .

Natural homomorphism from *R* to *R/A*

The mapping .

Homomorphism from *Z* to a Ring with Unity

Let *R* be a ring with unity 1. The mapping  given by  is a ring homomorphism.

A Ring with Unity Contains  or *Z*

If *R* is a ring with unity and the characteristic of *R* is , then *R* contains a subring isomorphic to . If the characteristic of *R* is 0, then *R* contains a subring isomorphic to *Z*.

 is a Homomorphic Image of *Z*

For any positive integer *m*, the mapping of  given by  is a ring homomorphism.

A Field Contains  or *Q*

If *F* is a field of characteristic *p*, then *F* contains a subfield isomorphic to . If *F* is a field of characteristic 0, then *F* contains a subfield isomorphic to the rational numbers.

Prime subfield

The smallest subfield (a subfield contained in every subfield).

Field of Quotients

Let *D* be an integral domain. Then there exists a field *F* (called the field of quotients of *D*) that contains a subring isomorphic to *D*.