Special need students school bus routing: Consideration for mixed load and heterogeneous fleet

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**ABSTRACT**

We consider the School Bus Routing Problem (SBRP) for routing special education students based on our experience at a large suburban school district in Western New York, United States. We found the problem to be significantly different from that of routing regular students. The principle differences include the need to pick up special education student from their home, the need to configure buses appropriately for special education students, and the need to provide a higher level of service. Building upon prior work we developed a greedy heuristic coupled with a column generation approach to obtain approximate solutions for benchmark instances. Our findings demonstrated a 10\textperthousand-20\textperthousand cost reduction, which is particularly significant since special education transportation account for 40\textperthousand of the transportation budget.

1. Introduction

The Individuals with Disabilities Education Act (IDEA) was originally enacted by the United States Congress in 1975 to ensure that children with disabilities have the opportunity to receive a free appropriate public education, just like other children. In IDEA, the term transportation embodies travel to and from school, between schools, and appropriate public education, just like other children. In IDEA, the term transportation embodies travel to and from school, between schools, and around school buildings. Specialized equipment, such as adapted buses, lifts, and ramps, is required when needed to provide the transportation. Also, transportation includes transit from house to bus; hence, when students are unable to get to school without assistance, door-to-door transportation is required. In addition to providing specialized equipment that is required to transport students safely, school boards even have to provide nurses or aides on vehicles if needed [1,2].

The additional requirements needed for transportation of special education students make this operation costly. During the 1999–2000 school year the U.S. total expenditure on special transportation services is estimated to be about $3.7 billion. This represents around 28\textperthousand of the total transportation expenditures ($13.1 billion) in the nation and approximately 7\textperthousand of the total spending on special education services ($50 billion) [3]. The transportation cost per regular education student is approximately $200 to $400, whereas the same for special education ranges from $4000 to $6000 per student; note that the actual cost depends on school schedules and district geography [1]. Since the transportation cost of special education students is significantly higher than the cost for the regular case, even small improvements can benefit school boards. In national studies, computer-generated routes have proved to be significantly (32\textperthousand) more efficient and cost-effective than hand-developed routes [1].

A significant difference between routing special education students as opposed to regular students is the diversity of the students and the programs they attend. In addition to special restrictions in travel time and equipment, special education students do not necessarily live close to their programs, whereas the most common School Bus Routing Problem (SBRP) for regular programs have a significant number of students living relatively close to their school. Moreover, the number of students per program is dramatically lower than for regular schools. In regards to the programs, these are geographically dispersed and have different start and end times. Their location is also particularly troubling since it is often beyond the limits of the school district, making the routes very long, which then allows the assignment of only a few students per bus. Thus, school districts often allow a mixed load configuration for these buses, where a bus can serve students from different schools. However, there is a lack of work in studying the mixed ride scenario for both special education and general students [4]. While more recent work [5–8] tackles mixed load for regular students with various considerations, no such advances have been made for special education students.

The most recent review of the SBRP states its research is extensive; however, the review clarifies only a few papers consider the problem of...
routing special education students [4]. To the best of our knowledge, only three papers have directly studied this problem [9-11]. Table 1 shows the main characteristics of each work, where the common theme is to focus on developing a heuristic solution method for a real-world inspired problem. Only in two of these papers the problem was modeled with a mathematical formulation, having that in neither case such formulation was used as part of the solution procedure.

The SBRP mainly consists of morning and afternoon problems. Many studies are dedicated to the morning problem, whereas the afternoon problem is only mentioned briefly [4], despite the fact they are different in their formulation and are equally complex. For example, the drop-off location for students in the afternoon may differ. Nowadays, many students do not return home after school; instead, they go to after school caretakers. Students with disabilities are entitled to be transported to caretakers even when those caretakers live out of a district’s attendance boundaries [2]. Also, schools do not necessarily have their time windows in the reverse sequence as in the morning. Thus, routes in the afternoon cannot be simply obtained as the reverse of the morning case.

In this paper, we study the SBRP for special education students inspired by our work with the Williamsville Central School District (WCSD), in Western New York, United States. The main contributions of this work are as follows:

- present a unified mathematical formulation for morning and afternoon routing problem with consideration for mixed loading, school time windows, and student maximum travel time;
- introduce a heterogeneous fleet, where the difference between buses is not only capacity but also the seat type configuration, combination of regular seat and wheelchair space; and
- present a numerical example along with real word instances.

2. Problem description

This research is a continuation our previous work [12] in the Williamsville Central School District (WCSD), the largest suburban school district in Western New York as part of their Transportation Operations Management Efficiency Program granted by the New York State Education Department. Our involvement in the Efficiency Program focused in the District’s Department of Transportation, where the program’s objective was to increase the efficiency of bus routing.

WCSD encompasses 40 square miles in the Buffalo metropolitan area, including parts of the towns of Amherst, Clarence, and Cheektowaga, enrolling over 10,000 kindergarten through 12th grade students. The problem studied in this research focuses on the transportation of students enrolled in Special Education programs. Though similar to the typical SBRP, there are significant differences that need careful consideration.

As in many school districts, transportation for special education students in WCSD is outsourced. However, the school district designs the routes and decides the number of buses to be utilized. This is done at least a month prior to the beginning of each school year. While the details of the agreement between the District and the third-party transportation company cannot be disclosed in this paper, we can mention that there is a full price for using one bus for both AM and PM runs during an entire semester, and a discounted price if a bus is only needed for either AM or PM runs. Also, additional charges apply in case the district requires more buses for non-scheduled events. Because the District does not own the buses, it has the liberty of choosing a fleet combination that fits the need for each year. A bus can be configured to hold up to three wheelchairs, having the remainder space setup with regular seats. Fig. 1 shows the layout of different seat type configurations, where that of 1a has only regular seats; 1b can hold only one wheelchair (in the back of the bus), 1c two wheelchairs, and 1d up to three. It is worth mentioning that for each configuration the bus capacity changes. Thus, the District not only decides the routes and the number of buses, but also the seat type configuration for those buses.

There are significantly fewer students in special education, about 4% of the total student population. However, the number of schools involved is much greater than for regular programs. They have different bell times and their locations are more diverse in comparison to the other schools, having several out of the district’s boundaries. While regular students are enrolled in schools depending on which school is the closest to their home address, special education students enroll in programs that meet their particular needs such as speech-language therapy, children’s psychiatric center, autism services, orthotic clinic, among others. As long as the location of the program is within the

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schools</td>
<td>Rusel 1986</td>
</tr>
<tr>
<td>Location type</td>
<td>Urban</td>
</tr>
<tr>
<td>Mixed load</td>
<td>Yes</td>
</tr>
<tr>
<td>Fleet</td>
<td>Homogenous</td>
</tr>
<tr>
<td>Objective</td>
<td>Minimize total</td>
</tr>
<tr>
<td>Constraints</td>
<td>Bus capacity</td>
</tr>
<tr>
<td>Problem size</td>
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</tr>
<tr>
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<tr>
<td>Solution method</td>
<td>Greedy heuristic</td>
</tr>
<tr>
<td>Reference</td>
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</tr>
<tr>
<td></td>
<td>Kamali 2013</td>
</tr>
<tr>
<td>Reference</td>
<td>Russell 1986</td>
</tr>
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<td>Ripplinger 2005</td>
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<td>Ripplinger 2005</td>
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<tr>
<td>Reference</td>
<td>Kamali 2013</td>
</tr>
</tbody>
</table>

Fig. 1. Bus layout for different seat type configurations.
Buffalo metropolitan area, the District will provide transportation to these students. Note that not all these special education programs are run by the state, but many are privately operated, and that each family freely chose the program that most fits the student’s need. When a student applies to a special education program run by the district, he or she is assigned a program depending on the student’s need and the program’s location, whereas when a family chose to have their child apply to a program not run by the district. Under both cases, the district would provide transportation upon request. From Fig. 2a and b we can see the significant difference between the locations of homes (in gray circles) and schools (in black squares) for the regular and special education cases for WCSD.

Provided that WCSD will send buses outside the district with underutilized capacity, other school districts could potentially share the cost of special education students and add their students to WCSD’s buses. A strategy like the one proposed by Ref. [13] could very well save considerable sums of tax payer’s money. However, New York State regulation prohibits this practice when the provider of the transportation service is a contractor [14], which is the case in our problem.

These particular features tend to make routes significantly longer [9] and with less use of bus capacity. Consequently, the operation of special education students is very expensive, reaching up to 40% of the annual transportation budget of WCSD. Thus, careful planning of this operation is needed to identify saving opportunities while guaranteeing the quality of service that these students require.

At WCSD, special education students are picked up at their doorstep whereas regular students are required to walk to a particular stop location, to which multiple students may be assigned. Additionally, every bus is obliged to have an aide on board to care for the students and assist those who require more attention; a nurse may also be required depending on the student’s needs. If needed, the buses have to be specially equipped, for example, they need to be able to handle safe transportation for students in wheelchairs, which reduces the capacity of the bus. We say a student has a demand for each seat type, equivalent to that of multimmodity problems. A bus may also carry students from different schools at the same time; hence, we modeled our problem considering mixed load.

Constraints associated with time consider time windows for morning drop-offs and afternoon pick-ups, default maximum ride time set to one hour that each student can change upon request. There are no restrictions on the hours that a bus can operate. Finally, each of the schools and students has a delay, i.e., service time, at drop-offs and pick-ups. This time is about 10 min for schools, and for pick-ups and drop-ups at the students’ home, the delay varies from 1 to 5 min depending on their needs.

While an important number of research has focused on the SBRP with some of the considerations needed for the case of WCSD [4–8,12], we have not been able to find work that tackles all of them for the case of special education students. Thus, in the following sections, we modeled and proposed a solution strategy for the problem at hand accompanied by numerical experiments.

3. Mathematical model

The following formulation considers the characteristics described in the previous section and supports school bus routing for both morning and afternoon runs. The objective of the model is to minimize the total number of buses used in the morning and afternoon runs. Later, we provide a solution strategy based on column generation. Even though our methods and testing are based on the Williamsville School District scenario, we believe that the methods can be readily modified for other school districts, and the results are widely applicable.

To support both AM and PM runs, we define $\theta$ as a binary parameter equal to 1 if routes are for the AM runs and 0 for the PM runs. In addition we define $\phi = 2(\theta - \frac{1}{2})$, i.e., equal to 1 and -1 for AM and PM runs respectively. These parameters enable us to formulate a unique model for both cases.

Consider the set $A$ of all stops for students, the set $S$ of all schools and the sets $D_1$ and $D_2$ corresponding to the start and end depots respectively. The set of all locations is $L$, with $L = D_1 \cup A \cup S \cup D_2$. Let the function $\delta(i)$ represent the school of the students in stop $i \in A$ (this means $\delta(i) \in S$). Let $t_i^k$ be the fixed waiting time at node $i \in A \cup S$ (time per student), $t_i^s$ be the variable waiting time at node $i \in A \cup S$ (time per student), and $\tau_i$ the maximum
travel time for student $i \in A$. Let $a_i$ and $b_i$ represent the earliest and latest time of arrival to location $i \in L$ (time window). For $i \in S$ the time windows of arrival $[a_i, b_i]$ for the rest of the assignment is as follows:

\[
\begin{align*}
    a_i = \begin{cases} 
        a_i - \theta (t_i + \tau_i + t_{ij}) + (1 - \theta)(t_j + t_{ij}) & \text{if } i \in A \\
        \min \{ a_i - t_j, (1 - \theta) \min_{j \in A} (a_i - t_j) \} & \text{if } i \in D_1 \\
        \min \{ a_i + t_j + t_{ij}, (1 - \theta) \min_{j \in A} (a_i + t_j + t_{ij}) \} & \text{if } i \in D_2 
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    b_i = \begin{cases} 
        b_i - \theta (t_i + t_j) + (1 - \theta) \max (t_i + t_j, t_{ij}) & \text{if } i \in A \\
        \max \{ b_i - t_j, (1 - \theta) \max (b_i - t_j) \} & \text{if } i \in D_1 \\
        \max \{ b_i + t_j + t_{ij}, (1 - \theta) \max (b_i + t_j + t_{ij}) \} & \text{if } i \in D_2
    \end{cases}
\end{align*}
\]

Let $G = (L, E)$ be a directed graph. The set of edges is $E = E_{DA} \cup E_{SS} \cup E_{AS} \cup E_{DA} \cup E_{SS} \cup E_{SD}$ where $E_{DA} = \{(i, j) \mid i \in D_1 \times A\}$ is the set of edges connecting the depot to the students, $E_{AS} = \{(i, j) \mid A \times S \} \neq (i, j), a_i + t_i + t_{ij} \leq b_i$ is the set of feasible links between students, $E_{SS} = \{(i, j) \mid S \times A \} \neq (i, j), a_i + t_i + t_{ij} \leq b_i$ is the set of feasible links between schools, and $E_{SD} = \{(i, j) \mid S \times D\}$ is the set of edges connecting schools to the depot.

Additionally, let $A_i = \{i \in A : \delta (i) = \theta (i)\}$ be the set of students attending school $j \in S$. Notice that the sets $A_i$ are mutually exclusive.

Regarding the attributes of the vehicles we consider in our model, we define $B$ to be the set of buses and $Q$ to be the set of seat types that the buses have (e.g., regular seats, wheelchair spaces). Let $d_{ij}$ be a binary parameter equal to 1 if bus $k \in B$ starts in depot $i \in D_1$ and finished $j \in D_2$, $s_{ij}$ be the number of seat types $q \in Q$ used by student $i \in A$, and $c_{qk}$ capacity of bus $k \in B$ in regards of seat-type $q \in Q$.

We now define the decision variables of our model. Let $z_i$ be a binary variable indicating if bus $k \in B$ is assigned to node $i \in L$ to node $j \in L$. Let $u_{ik}$ be the time of arrival of bus $k \in B$ at node $i \in L$, and $w_{ij}$ the load of bus $k \in B$ upon arrival at node $i \in L$. Thus, the formulation of the problem of finding the smallest fleet of buses while meeting the policies regarding service level reads as follows:

\[
\begin{align*}
    \min & \quad \sum_{k \in B} z_k + \varepsilon \sum_{i \in L, j \in \{ S \cup D \}} x_{ijk} \\
    \text{s.t.} & \quad \sum_{j \in \{ S \cup D \}} x_{ijk} = 1 & i \in A \\
    & \sum_{(l, j) \in E} x_{ijk} \leq \frac{1}{2} \sum_{k \in B} z_k & k \in B \\
    & \sum_{j \in \{ S \cup D \}} x_{ijk} \leq \max \{ n, d_{ijk} \} & i \in D_1, j \in B \\
    & \sum_{j \in \{ S \cup D \}} x_{ijk} \leq \min \{ n, d_{ijk} \} & i \in D_2, j \in B \\
    & \sum_{i \in \{ A \cup S \}} x_{ijk} \leq \sum_{i \in \{ A \cup S \}} x_{ijk} & j \in \{ A \cup S \}, k \in B \\
    & \sum_{q \in Q} s_{ij} \leq \sum_{k \in B} x_{ijk} & i \in A, k \in B \\
    & u_{ik} + t_i + t_{ij} \sum_{q \in Q} s_{ij} + t_{ij} \leq u_{jk} + M(1 - x_{ijk}) & (i, j) \in E \\
    & i \in L \setminus S, k \in B
    \end{align*}
\]

In Table 2 we show the results for model (1)–(15) when solving different size problems. We can see how rapidly the solving time increases. In particular for this test, we set a limit of 1 h for every instance. For the cases that reach the time limit, the optimality gap is presented.

A somewhat similar problem is the one presented by Ref. [5], where they formulate a school bus routing problem for regular students and mixed load as a pickup and delivery problem. To do so, they created a small size instances.

<table>
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<tr>
<th>Stops</th>
<th>Schools</th>
<th>CPU time</th>
<th>Initial Solution</th>
<th>Final Solution</th>
<th>Lower bound</th>
<th>Gap</th>
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<tr>
<td>5</td>
<td>5</td>
<td>1</td>
<td>2.5</td>
<td>2.43</td>
<td>2.43</td>
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<tr>
<td>10</td>
<td>6</td>
<td>11</td>
<td>3.5</td>
<td>3.48</td>
<td>3.48</td>
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</tr>
<tr>
<td>15</td>
<td>6</td>
<td>43</td>
<td>5.5</td>
<td>4.56</td>
<td>4.56</td>
<td>0%</td>
</tr>
<tr>
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<td>6</td>
<td>3603</td>
<td>6.3</td>
<td>5.52</td>
<td>5.45</td>
<td>1%</td>
</tr>
<tr>
<td>25</td>
<td>6</td>
<td>3607</td>
<td>6.5</td>
<td>6.48</td>
<td>3.58</td>
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</tr>
<tr>
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<td>3608</td>
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</tr>
<tr>
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</tr>
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</tr>
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<td>50</td>
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<td>3633</td>
<td>11.5</td>
<td>11.50</td>
<td>3.30</td>
<td>71%</td>
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delivery point for each student by replicating their corresponding schools. This formulation produces a total of \((2n)^2\) binary \(x_{ij}\) variables, where \(n\) is the number of students. However, in our formulation we generate \((n + m)^2\) binary \(x_{ij}\) variables, where \(n\) and \(m\) are the number of students and school respectively. Since \(m \ll n\) we have that \((n + m)^2 \ll (2n)^2\), i.e., our formulation contains fewer variables and constraints. Thus, we find that a contribution of model (1)–(15) is the offering of a formulation that more closely represents the problem at hand.

4. Solution strategy

Because of the complexity of the routing problem \(\mathcal{P}\) from the previous section, optimization packages like CPLEX are not able to optimally solve realistic size instances. Thus, we elect to solve the problem approximately. Our approach uses column generation within a procedure tailored for our problem's special characteristics. In the work of [12] a similar decomposition was successfully applied for the regular school bus routing problem. In this work we also use the concept of bus class and establish a subproblem for each one of them, and the decomposition is used heuristically as well. In addition, our subproblem is more complex, and we adopt a different approach to finding good solutions. The following elaborates on the aspects of this decomposition.

4.1. Problem decomposition by column generation

4.1.1. Master problem

Let \(P_k\) be the set of feasible paths for bus \(k \in B\), where \(p \in P_k\) is an elementary path. Let \(x_{ik}^p\) be equal to 1 if edge \((i, j) \in E\) is covered by bus \(k \in B\) when using path \(p \in P_k\). \(\psi^p_k = \sum_{i \in E} \sum_{j \in E} x_{ik}^p + \epsilon \sum_{i \in E} \sum_{j \in E} t_{ij} x_{ij}^p\) be the cost of using path \(p \in P_k\) with vehicle \(k \in B\) and \(\psi^p_k = \sum_{i \in E} \sum_{j \in E} x_{ij}^p\) be equal to 1 if stop \(i \in A\) is visited by bus \(k \in B\) when using path \(p \in P_k\) and 0 otherwise. Let \(\psi^p_k\) be the binary decision variables that are equal to 1 if path \(p \in P_k\) is used by bus \(k \in B\) and 0 otherwise. Then, the master problem reads as follows:

\[
\mathcal{P}_m: \min \sum_{k \in B} \sum_{p \in P_k} \psi^p_k y^p_k
\]

s.t. \(\sum_{k \in B} \sum_{p \in P_k} y^p_k \psi^p_k = 1, \quad i \in A\) \hspace{1cm} (16)

\(\sum_{p \in P_k} y^p_k \leq 1, \quad k \in B\) \hspace{1cm} (17)

\(y^p_k\) binary \hspace{1cm} (18)

Given that there are different seat configurations for the buses, we group the buses that share the same configuration into \(k\) bus classes. Therefore, let \(W\) define the set of bus classes, where each element \(w \in W\) represents a set of indistinguishable buses, and let \(K_w\) be the number of available buses for each class. Then, the new master problem reads as follows:

\[
\mathcal{P}_m: \min \sum_{w \in W} \sum_{p \in P_w} \psi^p_w y^p_w
\]

s.t. \(\sum_{p \in P_w} y^p_w \psi^p_w = 1, \quad i \in A\) \hspace{1cm} (20)

\(\sum_{p \in P_w} y^p_w \leq K_w, \quad w \in W\) \hspace{1cm} (22)

\(y^p_w\) binary \hspace{1cm} (23)

4.1.2. Subproblem

Since the variables or columns of the master problem represent paths for each of the bus classes, one subproblem must be solved for each bus class. Thus, there will be \(|W|\) subproblems to solve separately.

Let \(\pi_k\) represent the dual variables associated with constraints (21) and \(\rho_w\) represent the dual variables associated with constraints (22). Then, for a given bus \(k\) the subproblem minimizes the reduced cost \(\psi^p_k - \sum_{p \in P_k} \pi_k y^p_k + \rho_w\). Thus, the subproblem for class \(w \in W\) reads as follows:

\[
\mathcal{P}_w: \min \{1 - \rho_w + \sum_{(i,j) \in E} [\epsilon y_{ij} - \pi_k] x_{ij}\}
\]

s.t. \(\sum_{j \in L} x_{ij} \leq 1, \quad i \in A\) \hspace{1cm} (24)

\(\theta \sum_{j \in A} x_{ij} + (1 - \theta) \sum_{j \in S} x_{ij} = \sum_{j \in D_1} d_j, \quad i \in D_1\) \hspace{1cm} (25)

\(\delta \sum_{j \in A} x_{ij} + (1 - \delta) \sum_{j \in S} x_{ij} = \sum_{j \in D_2} d_j, \quad j \in D_2\) \hspace{1cm} (26)

\(\sum_{i \in L} x_{ij} = \sum_{j \in L} x_{ji}, \quad j \in A \cup S\) \hspace{1cm} (27)

\(\sum_{j \in L} x_{ij} \leq \sum_{j \in S} x_{ij}, \quad i \in A\) \hspace{1cm} (28)

\(u_i + t_i + \sum_{q \in Q} s_i^q + t_j \leq u_j + M_1(1 - x_{ij}), \quad (i, j) \in E: \quad i \in L \cup S\) \hspace{1cm} (29)

\(u_i + t_i + \sum_{q \in Q} s_i^q + t_j \leq u_j + M_1(1 - x_{ij}), \quad (i, j) \in E: \quad i \in S\) \hspace{1cm} (30)

\(0 \leq \varphi(u_i - u_j) - u_i \leq \max\{\tau, \beta h_s(j) + (1 - \theta) t_{ij}\}, \quad i \in A\) \hspace{1cm} (31)

\(a_i \leq u_i \leq b_i, \quad i \in L\) \hspace{1cm} (32)

\(u_i^q + \varphi s_i^q \leq v_i^q + M_2(1 - x_{ij}), \quad (i, j) \in E: \quad i \in L \cup S, \quad q \in Q\) \hspace{1cm} (33)

\(u_i^q - \varphi \sum_{q \in Q} s_i^q x_{ij} \leq v_i^q + M_1(1 - x_{ij}), \quad (i, j) \in E: \quad i \in S, \quad q \in Q\) \hspace{1cm} (34)

\(0 \leq v_i^q \leq c_i^q, \quad i \in L, \quad q \in Q\) \hspace{1cm} (35)

\(x_{ij} \in [0, 1], \quad (i, j) \in E\) \hspace{1cm} (36)

where (24) minimizes the reduced cost of the new variables for the master problem. Constraint (25) ensures that every student is considered at most once on any new path; recall that in the subproblem we are interested in finding single routes that do not necessarily need to contain all students. Constraints (26)–(37) are the single bus case of (4)–(15).

4.2. Column generation procedure

Our implementation of the column generation procedure follows the framework that Fig. 3 shows. In a general sense, the method contemplates the following four steps:

Step 1. Using a saving heuristic (see Section 4.2.1 for a description), we construct a set of initial solutions that is used to start the column generation process. We use the best solution from the initial set to establish the value of \(\epsilon\) as one tenth of the inverse of its total travel time; recall that \(\epsilon\) is a parameter in the objective functions (1) and (24).

Step 2. Solve the linear relaxation of the master problem with all the available columns. Update the coefficient of the objective function of the subproblems with the new values of the dual variables.

Step 3. Using a second saving heuristic (see Section 4.2.2 for a description), we generate new columns. If at least one of the generated columns in any of the subproblems has a negative cost, then go to
Step 2. Solve the master problem as an integer program using regular branch and bound.

In the following sections, we present further explanation on how the subroutines used here work.

4.2.1. Generating an initial solution

The procedure to generate the initial solution can be summarized as follows:

Step 1. Create a new route $R$.
Step 2. For every unassigned stop find, if feasible, the cheapest potential insertion into $R$.
Step 3. If at least one stop can be inserted into $R$, assign to $R$ one stop based on the saving cost that such insertion yields. If not, go to Step 1.
Step 4. If there are unassigned stops go to Step 2.

The algorithm creates a new route in Step 1 by first selecting a bus with the most availability of the seat type with least demand among the students that have not yet been assigned a route. Next, we filter the stops that have a positive demand for the selected seat type.

To choose which stop to add into the new route, for each stop, we calculate the resulting new cost of the route from adding such a stop. We will use Fig. 4 to illustrate Steps 2 and 3 of the procedure. Let us assume that we begin Step 2 with the provided current route, where two students have been assigned so far, and they both go to the same school. Three other students have not yet been assigned a route, and for each, we then evaluate their possible insertions. Student 3 has two possible options to be inserted into the route. Similarly, Student 4 has four possible options but note that this student goes to a different school, for which the algorithm needs to determine a position as well. We keep the cheapest option for each student as their potential insertion to the route. In the example, options 2, 3 and 1 are the most economical for students 3, 4 and 5 respectively.

In Step 3, the algorithm chooses one of the students based on the saving cost of their potential insertion. There are two ways to perform this operation: (1) deterministically, by choosing the students with the maximum saving cost, and (2) randomly, by selecting the student based on a probability proportional to $\left(\frac{S_i - S_{i-1}}{S_{i-1} - S_{i-2}}\right)$ where $i$ represents the student, $\gamma = 2$ and $S[i]$ is the student’s saving cost previously calculated. In the example, we chose student 4, and we then repeat the process for the modified route. In the case where the algorithm cannot find any feasible insertion for any of the unassigned students, we stop working with the current route and create a new one in Step 1.

This algorithm returns a feasible solution for our problem: a set of routes where every student has been assigned to one and only one bus. To create diversity in the set of initial solutions, we run our implementation of the algorithm once in deterministic mode and fifty times in random mode (recall Step 3, where the next insertion can be randomly selected). Then, we pass the set of initial solutions to the master problem to continue with the column generation procedure.

4.2.2. Approximate solution for the subproblem

The subproblem is NP-hard. Therefore, finding an optimal solution for it is computationally intensive. Thus, we chose to solve the subproblems approximately by means of a heuristic summarized as follows.

Step 1. Create a new route $R$ for the given bus class.
Step 2. For every unassigned stop find, if feasible, the cheapest potential insertion into $R$.
Step 3. If at least one stop can be inserted into $R$, assign to $R$ one stop based on the saving cost that such insertion yields. If not,
The algorithm creates an empty route using a bus from the given class (recall that there is a subproblem for each bus class). Steps 2 and 3 of this algorithm are fundamentally the same as in the procedure explained in the previous section. Thus, the example from Fig. 4 also illustrates the subroutine in this algorithm.

This algorithm returns a single route for the given bus class. To accelerate the column generation procedure, our implementation runs the algorithm once in deterministic mode and twenty times in random mode for each subproblem. All the solutions generated with negative objective value are then passed to the master problem to continue with the column generation procedure.

4.3. A lower bound

A simple way to find a lower bound of problem $\mathcal{P}_i$ is to solve its linear relaxation, i.e., exchange constraint (15) for the inequality $0 \leq x_{i,k} \leq 1$. However, this bound is weak and is computationally costly to obtain. Alternatively, a column generation procedure can be used for the same purpose. In the preceding Section we described a decomposition by column generation where the subproblem is solved approximately due to its complexity; hence, it does not yield a valid lower bound. Then, we hereby present an alternate formulation of $\mathcal{P}_i$ that allows for an easy and quick way of obtaining a lower bound.

Let $z_k$ be a binary variable indicating if bus $k \in B$ is used, and $x_{i,k}$ be a binary variable indicating if node $i \in A \cup S$ is visited by bus $k \in B$. For any non-empty set of students $G \subset A$ let $r(G) = |G|$ if a feasible route containing all nodes in $G$ exists, and let $r(G) = |G| - 1$ otherwise. Consider the following formulation:

$$\mathcal{P}_i: \min \sum_{k \in B} z_k$$  \quad (38)

s.t. $\sum_{i \in A \cup S} x_{i,k} \leq M_k z_k \quad k \in B$  \quad (39)

$\sum_{i \in G} x_{i,k} \leq M_k x_{i,k} \quad k \in B, j \in S$  \quad (40)

$x_{i,k} = 1 \quad i \in A$  \quad (41)

$\sum_{i \in G} x_{i,k} \leq r(G) \quad k \in B, G \subset A$  \quad (42)

$x_{i,k} \in [0,1] \quad k \in B, i \in A \cup S$  \quad (43)

$z_k \in [0,1] \quad k \in B$  \quad (44)

where the objective (38) is to minimize the number of buses, constraints (39) ensure only buses being used can visit any node, (40) only allow assignment of students to buses that visit their corresponding school, (41) ensures every student is assigned a bus, and (42) cut infeasible solutions.

Notice $\mathcal{P}_i$ is a compact formulation of $\mathcal{P}_i$. Hence, we now focus on finding a lower bound for $\mathcal{P}_i$. Also, notice the number of constraints represented by (42) increases exponentially with the number of students. To handle this, we leave out all constraints in (42) gives a relaxation of $\mathcal{P}_i$ and then we proceed to solve with cutting-planes, where at each iteration we add the corresponding cut in (42) if an infeasible solution is found. Since this method yields a lower bound at every iteration, we can stop at any time. However, at the beginning of the algorithm the value of the lower bound is very weak, as can be expected since constraints (39)-(41) contain almost no information about the feasibility of the routes.

To strengthen the lower bound we now modify these constraints. Let $Z = \{(i, j) \in N^2: i < j, (i, j) \notin E, (j, i) \notin E\}$ be the set of conflicting rides, where any element in this set represents two nodes that cannot be visited by the same vehicle. Let $\hat{t}_k = t_i + \min_{(k,i)}[\hat{t}_k]$ be the minimum amount of time spent when visiting node $i \in A \cup S$. Let $b(i)$ be the set of buses that student $i \in A$ can ride. Consider the following formulation $\mathcal{P}_i$: min $\sum_{k \in B} z_k$  \quad (45)

s.t. $x_{i,k} + x_{j,k} \leq z_k \quad (i, j) \in Z, k \in B$  \quad (46)

$\sum_{j \in G} x_{i,k} \leq \max[b(i) + \hat{t}_k] - \min[a_i + \hat{t}_k] z_k \quad k \in B$  \quad (47)

$\sum_{j \in G} a_j x_{j,k} \leq c_{i,k} x_{i,k} \quad k \in B, q \in Q, j \in S$  \quad (48)

$x_{i,k} = 1 \quad i \in A$  \quad (49)

$\sum_{k \in b(i)} x_{i,k} = 0 \quad i \in A$  \quad (50)

$\sum_{k \in G} x_{i,k} \leq r(G) \quad k \in B, G \subset A$  \quad (51)

$x_{i,k} \in [0,1] \quad k \in B, i \in A \cup S$  \quad (52)

$z_k \in [0,1] \quad k \in B$  \quad (53)

where the objective (45) minimizes the number of buses, (46) prohibits having in the same bus two students that cannot ride together, (47) is a proxy of travel time constraints, (48) only allow assignment of students to buses that visit their corresponding school and also act as a proxy of capacity constraints. (49) and (50) ensures that students are assigned in buses that they are allowed to ride. Finally, (51) is equivalent to (42). Constraints (47) and (48) are the modified version of (39) and (40) respectively, and (49) and (50) modify constraints (41). Notice that the feasible space of $\mathcal{P}_i$ is more constrained than that for $\mathcal{P}_i$. As a result we can obtain a stronger lower bound for $\mathcal{P}_i$.

5. Computational experiments

5.1. Simulated data

To understand the influence of some characteristics of the problem on its objective function, we designed a factorial experiment with five factors, where the response is the number of buses needed. The first factor is "Bell Time Offset", defined as the difference in time of the start time of any two groups of schools. The four levels of this factor are 0, 10, 20 and 30 min. The second factor is "Load Type", with two levels: mixed and single. Recall that the problem allows for a mixed ride, i.e., having students of different schools on the same bus. The single load type represents the case where only students of the same school can share a bus. The third factor is the maximum distance to any school from an arbitrary center, with five levels: 2.5, 5, 7.5, 10 and 12.5 miles. The fourth factor is the bus capacity, which was simply set to two levels: 5 and 10 seats. Finally, the fifth factor is the maximum ride time with two levels: 45 and 60 min.

In our experiment, we randomly generated 100 student locations and 20 schools around a single point. The students are randomly located within 5 miles of the center point, and the schools are located within different levels of distance (2.5, 5, 7.5, 10 and 12.5 miles) from the center depending on the experiment.

Fig. 5 shows the result of the numerical experiment. We can see that all the five factors influence the number of buses needed, having the bell time offset with the highest influence. The greater the offset, the fewer buses are needed. Having wider spacing between the start time of the schools allows better re-utilization of the buses. In this experiment, we observed an average reduction of 52% when varying the bell time...
offset from zero to half an hour.

Regarding capacity, as expected, the more seats available, the fewer buses are needed. The school location also affects fleet size; the further the schools are allowed to be located the more buses are needed.

The load type, more often than not, influences the number of buses. A mixed ride policy allows for the need of fewer buses in most cases. This is especially true for rather narrow bell-time offsets; we can see the biggest difference of the policy is for an offset of 10 min. Let us assume that there are only 10 min between the start time of two schools. A bus wouldn’t be able to pick up students after the first school to be able to drop them off at the second school. Therefore, to be able to reuse that bus for a second school, students for both schools would need to be picked up before visiting the first school, i.e., a mixed ride policy is appropriate. Now, if more time exists between start times, there will be sufficient time to pick up students of the second school after visiting the first one. Therefore, a combined policy of mixed ride and wider bell time offsets is desirable for minimizing the need for buses.

5.2. Case study

We hereby consider four real-world instances from WCSD, corresponding to AM and PM runs of the 2013–2014 and 2014–2015 school years. Recall that the geographical dispersion of these instances follows the nature seen in Fig. 2b. We run each of these instances for each combination of the factors “Maximum Ride Time” and “Load Type”, resulting in a total of 24 numerical experiments. For the maximum ride time we considered three levels: 40, 50 and 60 min. For the load type there are two levels: mixed and single. The results are shown in Table 3.

For each instance we show the name, the number of stops (students), schools, and maximum ride time allowed in minutes. Under column Edges we show the total number of edges and valid edges after discarding those that do not conform to the set defined as a function of
the time windows with the procedure described in Section 3, under CPU time we showed the computational time needed to find the initial solution and the time needed to run the column generation procedure (both times are in seconds). Under Buses we show the number of vehicles in the solution for each one of the methods utilized. The lower bound for each instance is shown under LB and it was obtained following the method described in Section 4.3. Column CG presents the number of vehicles found with the heuristic presented in Section 4.2.1 that finds a set of initial solutions for the column generation procedure. Under column Current, we show the final result for our procedure, i.e., the result obtained at the end of the column generation. Column Current buses shows the real number used at WCSD. This amount is only available for instances with 60 min of maximum ride time. Finally, column Buses saved shows the difference between the current situation and the solution found after using column generation. Notice that the current situation is available for four instances with 1 h of maximum ride time and a mixed ride policy.

By comparing the load type we can see the benefit of implementing the “Mixed Ride” over the “Single Ride” policy. The former requires, on average, 4% fewer buses than the latter. Also, in Fig. 6 we can see some interaction between the maximum ride time and the load type. The effect of the load type policy increases when the maximum ride is 60 min; if students can ride for a longer time in a bus, then there is more time to go around picking students up from different schools. Regarding computational complexity, the single ride policy results in a smaller problem, evidenced by the lower number of edges (under valid edges), requiring less time to solve.

In addition, we compare our results with the current situation at the school district for the corresponding school year. We can see that in all of the instances our solution outperforms the current practice, saving an average 20% of buses in a reasonable computational time. Recall that our approach takes advantage of the mixed load strategy for building the initial solution and further improves it via column generation. We can see that the column generation procedure improves the initial solution in almost every instance in an average of 6%.

6. Conclusion

In this work, we modeled and presented a solution scheme for the mixed load routing problem for special education students. Given the nature of dispersed locations of the stops and schools for this class of problem, we took advantage of the mixed ride strategy that allows a bus to carries students attending different schools and modeled the problem accordingly.

Special education students may require specially equipped buses for wheelchairs. Therefore, our model supports the use of a heterogeneous bus fleet. To find the appropriate number of regular and specially equipped buses to be leased, we start with an overestimated number of buses available to solve the problem and let the optimization choose the best number for each class of buses.

In addition, our formulation responds to both morning and afternoon problems in a uniform way. Provided that for this problem we allow mixed ride and that the schools have different start and end times, it is not appropriate to use the same sequence of stops for the morning in the afternoon as suggested in previous work. See Ref. [4] for a discussion in the morning versus afternoon problems.

Our numerical experiment demonstrates the benefits of using a combined policy of mixed ride and wider bell time offsets: fewer buses are needed when students from different schools can ride the same bus simultaneously and when schools have significantly different start

Table 3
Computational results for real instances of WCSD.

<table>
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<tr>
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<th>Stops</th>
<th>Schools</th>
<th>Bus Max</th>
<th>Load Edges</th>
<th>PC time</th>
<th>Buses Current</th>
<th>Buses saved</th>
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Fig. 6. Average number of buses per Load Type and Maximum Ride Time.
times. In this regards, two other findings are worth noting. First, by only looking at the bell time offset we observe a reduction by half of the fleet size when varying from zero to 30 min. Second, the biggest reduction of the fleet size explained by the mixed ride policy is observed when the bell time offset is 10 min, a rather modest level of such factor; mixed load allows having students from different schools picked up before visiting two schools 10 min apart.

By using a customized column generation procedure, we approximately solved a set of instances from a real school district in Western New York. We found that our approach responds well to cases with high dispersion of student and school locations. One reason for this is that our approach builds the routes allowing mixed ride from the beginning, whereas [5]’s approach is to improve a set of routes initially created with a homogeneous load. Validated by instances in a real school district, our proposed approach could save an average of 20% buses for special education students as compared to the existing bus operations.

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References


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