Centroids of Volumes and Lines

- It can be defined as average position of volumes and lines.
- This concept is used to compute centers of mass of certain types of object, i.e., where the weight effectively acts.

VOLUMES

Consider a volume \( V \) and let \( dV \) be a differential element of \( V \) with coordinates \( x, y, z \). By analogy of centroid of areas,

\[
\bar{x} = \frac{\int_{V} x \, dV}{\int_{V} dV}, \quad \bar{y} = \frac{\int_{V} y \, dV}{\int_{V} dV}
\]

\[
\bar{z} = \frac{\int_{V} z \, dV}{\int_{V} dV}
\]
LINES

Let \( dl \) be differential length of line \( L \) with coordinates \( x, y, z \).

The coordinates of the centroid of line \( L \) are

\[
\bar{x} = \frac{\int_{L} x \, dl}{\int_{L} dl},
\]

\[
\bar{y} = \frac{\int_{L} y \, dl}{\int_{L} dl},
\]

\[
\bar{z} = \frac{\int_{L} z \, dl}{\int_{L} dl}.
\]
Determine centroid of a cone by integration.

\[
\bar{x} = \frac{\int x \, dV}{\int dV}
\]
From similar triangles:

\[ \frac{p}{x} = \frac{R}{h} \]

\[ \varphi = \left( \frac{R}{h} \right) x \]

\[ dV = \pi \varphi^2 = \pi \left( \frac{R}{h} x \right)^2 = \pi \frac{R^2 x^2}{h^2} \]

\[ -x = \frac{\int_V x \, dV}{\int_V dV} = \frac{\int_0^h x \left( \pi \frac{R^2 x^2}{h^2} \right) \, dx}{\int_0^h \frac{\pi R^2 x^2}{h^2} \, dx} \]
\[ \bar{x} = \frac{\pi R^2}{h^2} \int_0^h x^3 \, dx \]
The line $L$ is defined by the function $y = x^2$. Determine the $x$ coordinate of its centroid.

\[ \bar{x} = \frac{\int_L x \, dL}{\int dL} \]

\[ dL^2 = dx^2 + dy^2 \]

\[ dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \]

\[ y = x^2 \text{ is given} \]

\[ \frac{dy}{dx} = 2x \]

Plug (2) in (1)
\[ dL = \sqrt{1 + 4x^2} \, dx \]

Plug the above in (*)

To integrate over entire volume, we should integrate from \( x = 0 \) to \( x = 1 \)

\[
\bar{x} = \frac{\int_0^1 x \, x \, dL}{\int_0^1 dL} = \frac{\int_0^1 x \sqrt{1 + 4x^2} \, dx}{\int_0^1 \sqrt{1 + 4x^2} \, dx}
\]

\[ = 0.574 \]

Centroids of Composite Volumes and Lines

Using similar approach as that applied to areas, the coordinates of the centroid of composite volumes are:

\[
\bar{x} = \frac{\sum_i x_i \, V_i}{\sum_i V_i}, \quad \bar{y} = \frac{\sum_i y_i \, V_i}{\sum_i V_i}, \quad \bar{z} = \frac{\sum_i z_i \, V_i}{\sum_i V_i}
\]
The coordinates of the centroid of a composite line are

\[
\bar{x} = \frac{\sum x_i L_i}{\sum L_i}, \quad \bar{y} = \frac{\sum y_i L_i}{\sum L_i}, \quad \bar{z} = \frac{\sum z_i L_i}{\sum L_i}
\]

To determine the centroid of composite volume or lines:

(a) Choose the parts
(b) Determine the value of parts
(c) Calculate centroid
1. Choose the parts: cone + cylinder

2. Determine the values for the parts using information from Appendix C

<table>
<thead>
<tr>
<th>Part</th>
<th>$x_i$</th>
<th>$V_i$</th>
<th>$x_i V_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part 1 (cone)</td>
<td>$\frac{3}{4} h$</td>
<td>$\frac{1}{3} \pi R^2 h$</td>
<td>$(\frac{3}{4} h) \left(\frac{1}{3} \pi R^2 h\right)$</td>
</tr>
<tr>
<td>Part 2 (cylinder)</td>
<td>$h + \frac{1}{2} b$</td>
<td>$\pi R^2 b$</td>
<td>$(h + \frac{1}{2} b) \left(\pi R^2 b\right)$</td>
</tr>
</tbody>
</table>
Calculate Centroid

\[ \bar{x} = \frac{x_1 V_1 + x_2 V_2}{V_1 + V_2} \]

\[ = \left( \frac{3}{4} h \right) \left( \frac{1}{3} \pi R^2 h \right) + \left( h + \frac{1}{2} b \right) \left( \pi R^2 b \right) \]

\[ \frac{1}{3} \pi R^2 h + \pi R^2 b \]

Because of symmetry \( \bar{y} = 0 \) and \( \bar{z} = 0 \).

**Pappus-Guldinus Theorems**: Two simple and useful theorems relating surfaces and volumes of revolution to centroids of the lines and areas that generate them.
Proof

The area \( dA \) is generated by element \( dL \) of the line and is given by:

\[
dA = (2\pi y)(dL)
\]

\[
A = 2\pi \int y dL \quad \text{-- 1}
\]

From definition of the y-coordinate of the centroid of a line,

\[
\bar{y} = \frac{\int y dL}{\int dL} \implies \int y dL = \bar{y} L \quad \text{-- 2}
\]

Put \((2) \text{ into (1)}\)

\[
A = 2\pi \bar{y} L
\]
Second Theorem

Consider an area in x-y plane that does not intersect the x-axis.

Let coordinates of the centroid of the area be \( \bar{x}, \bar{y} \).

Generate a volume by revolving the area about the x-axis.

The second Pappus–Guldinus theorem states that the volume \( V \) of the revolution is equal to the product of distance through which the centroid of the area moves and the area:

\[
V = 2\pi \bar{y} A
\]
(4) Use the Pappus-Guldinus to determine the surface area $A$ and volume $V$ of the given cone.

Revolving this straight line about $x$-axis generates the curved surface of cone.

- The centroid of the cone is $\bar{y} = \frac{1}{2} R \quad (1)$
- Length is $L = \sqrt{l^2 + R^2} \quad (2)$
- Centroid of the cone moves a distance $2\pi R \bar{y} L \quad (3)$