Centroid represents geometric center of a body.

**Centroid of Areas**

Consider an arbitrary area $A$ in $x$-$y$ plane.

Divide area into $A_1, A_2, \ldots, A_n$ and denote position of the parts by $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Centroid is then,

$$
\overline{x} = \frac{\sum x_i A_i}{\sum A_i}, \quad \overline{y} = \frac{\sum y_i A_i}{\sum A_i}
$$

In integral form,

$$
\overline{x} = \frac{\int_A x \, dA}{\int_A dA}, \quad \overline{y} = \frac{\int_A y \, dA}{\int_A dA}
$$
Locate the centroid of rectangular area shown in the figure below.

Solution

\[ y_c = \frac{\int y \, dA}{A} \quad \text{and} \quad x_c = \frac{\int x \, dA}{A} \]

\[ \int_A y \, dA = \int_0^h \int_0^b y \, dy \, dx = b \int_0^h y \, dy = b \left[ \frac{y^2}{2} \right]_0^h = \frac{bh^2}{2} \]
\[ x_c = \frac{\int x \, dA}{A} \]
\[ \int_A x \, dA = \int_0^b x (h \, dx) = h \left[ \frac{x^2}{2} \right]_0^b \]
\[ = \frac{hb^2}{2} \]

\[ y_c = \frac{\int_A y \, dA}{A} = \frac{bh^2}{2h} = \frac{h}{2} \]

\[ x_c = \frac{\int_A x \, dA}{A} = \frac{hb^2}{2bh} = \frac{b}{2} \]

\[ x_{c1} y_{c1} = \left( \frac{b}{2}, \frac{h}{2} \right) \]
Centroid of Composite Areas

If an area consists of combination of simple areas or a composite area, an easy approach can be used.

We can determine the centroid of a composite area without integration if the centroids of its parts is known.

\[
\bar{x} = \frac{\int_A x \, dA}{\int_A dA} = \frac{\int_{A_1} x \, dA + \int_{A_2} x \, dA + \int_{A_3} x \, dA}{\int_{A_1} dA + \int_{A_2} dA + \int_{A_3} dA}
\]

Let,

\[
\bar{x}_1 = \frac{\int_{A_1} x \, dA}{\int_{A_1} dA}
\]
\[ \int_{A_1} x \, dA = \bar{x}_1 A_1 \]

Similarly,
\[ \int_{A_2} x \, dA = \bar{x}_2 A_2 \quad \text{and} \quad \int_{A_3} x \, dA = \bar{x}_3 A_3 \]

Putting all of the three in (1)

\[
\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3}{A_1 + A_2 + A_3}
\]

Centroid of a composite area with arbitrary number of parts,

\[
\bar{x} = \frac{\sum_i x_i A_i}{\sum_i A_i}, \quad \bar{y} = \frac{\sum_i y_i A_i}{\sum_i A_i}
\]

Centroid of simple area is tabulated in the book.
Suppose you have a cut geometry.

The centroid of composite area,

\[
\bar{x} = \frac{\int_{x_1}^{x_2} x \, dA - \int_{x_1}^{x_2} x \, dA}{\int_{A_1}^{A_2} dA - \int_{A_1}^{A_2} dA} = \frac{x_1 A_1 - x_2 A_2}{A_1 - A_2}
\]
To calculate centroid of composite area following steps are required:

(i) Choose the parts—Try to divide the composite area into parts whose centroids you know.

(ii) Determine the centroid and areas of each part

(iii) Calculate centroid of the whole geometry using (iii)

2. Determine the centroid of the area shown
1. Choose Parts: Triangle, Rectangle, Semicircle

2. Determine the Values for the parts.

<table>
<thead>
<tr>
<th>Part</th>
<th>$\bar{x}_i$</th>
<th>$A_i$</th>
<th>$\bar{x}_i A_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part 1</td>
<td>$\frac{3}{2}b$</td>
<td>$V_2 b (2R)$</td>
<td>$\left(\frac{2}{3}b\right) \left[\frac{1}{2}b(2R)\right]$</td>
</tr>
<tr>
<td>Part 2</td>
<td>$b + \frac{1}{2}c$</td>
<td>$C(2R)$</td>
<td>$(b + \frac{1}{2}c)(C(2R))$</td>
</tr>
<tr>
<td>Part 3</td>
<td>$b + c + \frac{4R}{3\pi}$</td>
<td>$\frac{1}{2} \pi R^2$</td>
<td>$(b + c + \frac{4R}{3\pi})(\frac{1}{2} \pi R^2)$</td>
</tr>
</tbody>
</table>

3. Calculate the Centroid.

$$
\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3}{A_1 + A_2 + A_3}
$$
\[ \bar{x} = \left( \frac{2}{3} b \right) \left[ \frac{1}{2} b(2R) \right] + \left( b + \frac{1}{2} \right) [c(2R)] + \left[ b + c + \frac{4R}{3\pi} \right] \left( \frac{1}{2} \pi R^2 \right) \]

\[ \frac{1}{2} b(2R) + c(2R) + \frac{1}{2} \pi R^2 \]

Similarly, for \( \bar{y} \):

2) Determine the value for parts

<table>
<thead>
<tr>
<th>Part</th>
<th>( \bar{y}_i, A_i )</th>
<th>( \bar{y}_i, A_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part 1</td>
<td>( \frac{1}{3} (2R) )</td>
<td>( \frac{1}{2} b(2R) )</td>
</tr>
<tr>
<td>Part 2</td>
<td>( R )</td>
<td>( c(2R) )</td>
</tr>
<tr>
<td>Part 3</td>
<td>( R )</td>
<td>( \frac{1}{2} \pi R^2 )</td>
</tr>
</tbody>
</table>

3) Calculate the Centroid

\[ \bar{y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3}{A_1 + A_2 + A_3} = \frac{\left[ \frac{1}{3} (2R) \right] \left[ \frac{1}{2} b(2R) \right] + R \left[ c(2R) \right] + R \left( \frac{1}{2} \right) \pi}{\frac{1}{2} b(2R) + c(2R) + \frac{1}{2} \pi R^2} \]
To describe the load, a function $w$ is defined such that the downward force exerted on an infinitesimal element $dx$ of the beam is $wdx$.

With this function we can model varying magnitude of the load.

Arrows indicate that the load acts in downward direction.

SI units for $w$: Newton per meter.
Determining Force and Moment

The graph of w is called the loading curve.

Since the force acting on an element dx of the line is wdx, total force F exerted by distributed load can be found by integrating the loading curve with respect to x.

\[ F = \int w \, dx \]
Total moment about origin due to the force exerted on the element $dx$ is $x \cdot w \cdot dx$.

The total moment about the origin due to the distributed load is:

$$M = \int_{-L}^{L} x \cdot w \cdot dx.$$

For equivalence, force must act at position $x$ on the $x$-axis such that moment of $F$ about origin is equal to the moment of the distributed load about the origin:

$$\overline{x} \cdot F = \int_{-L}^{L} x \cdot w \cdot dx.$$

Therefore $F$ is equivalent to distributed load if we place it at the position:

$$\overline{x} = \frac{\int_{-L}^{L} x \cdot w \cdot dx}{F} = \frac{\int_{-L}^{L} x \cdot W \cdot dx}{\int_{-L}^{L} w \cdot dx} = 0.$$
Consider the following loading curve:

\[ y \]

\[ x \]

\[ F = A \]

\[ dA = wdx \]

\[ \int_{L} wdx = \int_{A} dA = A. \]

Plug \[ wdx = dA \] in (2):

\[ x = \frac{\int_{L} x wdx}{\int_{L} wdx} = \frac{\int_{A} wdx}{\int_{A} dA}. \]
The force \( F \) is equivalent to distributed load if it acts at the centroid of the "area" between the loading curve and \( x \)-axis.

The beam above is subjected to a "triangular" distributed load whose value at \( B \) is 100 N/m.

(a) Represent the distributed load by a single equivalent force.

(b) Determine reactions at \( A \) & \( B \).

Using area analogy, the equivalent force is:

\[
\frac{1}{2} \text{ base} \times \text{height} = \frac{1}{2} \times 12 \times 100 = 600 \text{ N}
\]

This acts at the centroid:

\[
\bar{x} = \frac{2}{3} (12) = 8 \text{ m}
\]
From equilibrium equations:

\[ \sum F_x = A_x = 0 \]
\[ \sum F_y = A_y + B - 600 = 0 \]
\[ \sum M_A = 12B - 8(600) = 0 \]

\[ A_x = 0 \quad A_y = 200 \text{ N} \]
\[ B = 400 \text{ N} \]