Parallel-axis Theorems

Objective: 1) To compute moment of inertia in another coordinate system given its value in terms of a different coordinate system.

2) It can be used to compute moment of inertia of a composite area.

Suppose moment of inertia of area $A$ in terms of area $A$ in terms of coordinate system $x'y'$ with its origin at the centroid of the area $y'y'$ coordinate system by calling $y'$ and

To compute moment of inertia of a parallel coordinate system $x'y'$ let centroid of $A$ in $x'y'$ coordinate be $(d_x, d_y)$ & $d = \sqrt{d_x^2 + d_y^2}$

In terms of $x'y'$ coordinate system the coordinate of centroid of $A$ are

\[
\bar{x}' = \frac{\int_A x'y'dA}{\int_A dA}, \quad \bar{y}' = \frac{\int_A y'y'dA}{\int_A dA}
\]

But the origin of $x'y'$ coordinate system is located at centroid of $A$ so $\bar{x}' = 0$ & $\bar{y}' = 0$
\[ \int_A x' \, dA = 0 \quad \text{and} \quad \int_A y' \, dA = 0 \]

**Moment of Inertia about the x-axis:**

In terms of \( xy \) coordinate system:
\[ I_x = \int_A y'^2 \, dA \]

Let \( y = y' + dy \) be coordinate of \( dA \) relative to \( xy' \) coordinate system.

Plugging this in above,
\[ I_x = \int_A (y' + dy)^2 \, dA = \int_A \left( (y')^2 + (dy)^2 + 2(y')(dy) \right) \, dA \]

\[ = \int_A (y')^2 \, dA + 2dy \int_A y' \, dA + d^2y \int_A dA \]

Moment of inertia of \( A \) about \( x' \)-axis

\[ \therefore I_x = I_{x'} + d^2 y A \]

This is a parallel axis theorem, it relates moment of inertia of \( A \) about \( x' \)-axis through the centroid to moment of inertia about parallel axis \( x \).
Similarly, moment of inertia about y-axis:

\[ I_y = I_y' + d_x^2 A \]

Product of inertia:

\[ I_{xy} = I_{xy}' + d_x d_y A \]

Polar moment of inertia:

\[ J_0 = J_0' + (d_x^2 + d_y^2) A \]

\[ J_0 = J_0' + d^2 A \]

You can also use the above ideas to compute moment of inertia of composite objects.

Determine moment of inertia of the given composite area.

[Diagram of a composite area with dimensions 4m x 3m and 2m x 1m]
Introduce coordinate system $x'y'$ with origin at centroid.

Moment of inertia of rectangular parts in terms of these coordinate systems in Appendix A.

To compute moment of inertia about $x$ axis:

$$I_x = I_{x'} + dy^2$$

<table>
<thead>
<tr>
<th>Part</th>
<th>$dy$</th>
<th>$A$</th>
<th>$I_{x'}$</th>
<th>$I_x = I_{x'} + dy^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part 1</td>
<td>2</td>
<td>(1) (4)</td>
<td>$\frac{1}{12}(1)(4)^3$</td>
<td>21.33</td>
</tr>
<tr>
<td>Part 2</td>
<td>0.5</td>
<td>(2) (1)</td>
<td>$\frac{1}{12}(2)(1)^3$</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Moment of inertia of the composite area about the $x$ axis is

$$I_x = (I_x)_1 + (I_x)_2 = 21.33 + 0.67 = 22 \text{ m}^4$$
Rotated and Principal Axes

Many engineering applications require computing of moments of inertia of areas with various angular orientations relative to coordinate system and also determine the orientation for which the value of moment of inertia is maximum or minimum.

Rotated Axes

Consider A in the coordinate system xy and a second coordinate system x'y' that is rotated through an angle θ relative to xy.

Goal: To compute moment of inertia in the terms of x'y'.

Let r be radial distance to differential element dA and the angle x, the coordinates of dA in xy coordinate system:

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
The coordinates of \( da \) in \( x'y' \) system are:

\[
x' = r \cos(x - \theta) = r \cos(x \cos \theta + \sin x \sin \theta) \\
y' = r \sin(x - \theta) = r(\sin x \cos \theta - \cos x \sin \theta)
\]

\[
x' = r \cos x \cos \theta + r \sin x \sin \theta \\
y' = r \sin x \cos \theta - r \cos x \sin \theta
\]

Using the value from (1):

\[
\begin{align*}
x' &= x \cos \theta + y \sin \theta \\
y' &= -x \sin \theta + y \cos \theta
\end{align*}
\]

**Moment of Inertia about \( x' \) axis**

\[
I_{x'} = \int_A (y')^2 \, dA = \int_A (-x \sin \theta + y \cos \theta)^2 \, dA
\]

\[
= \int_A y^2 \cos^2 \theta \, dA - 2xy \sin \theta \cos \theta + \sin^2 \theta x^2 \, dA
\]

\[
= \cos^2 \theta \int_A y^2 \, dA - 2 \sin \theta \cos \theta \int_A xy \, dA + \sin^2 \theta \int_A x^2 \, dA
\]

\[
I_{x'} = I_x \cos^2 \theta - 2I_{xy} \sin \theta \cos \theta + I_y \sin^2 \theta
\]

(2)

**Similarly, Moment of Inertia about \( y' \) axis**

\[
I_{y'} = I_x \sin^2 \theta + 2I_{xy} \sin \theta \cos \theta + I_y \sin^2 \theta
\]

(3)
Product of Inertia

$$I_{x'y'} = (I_x - I_y) \sin \theta \cos \theta + (\cos^2 \theta - \sin^2 \theta)I_{xy} \quad \rightarrow (4)$$

Polar Moment of Inertia

$$J'_0 = I'_{x'} + I'_{y'} = I_x + I_y = J_0 \quad \rightarrow (5)$$

Principal Axes

A rotated coordinate system $x'y'$ that is oriented so that $I_{x'}$ and $I_{y'}$ have maximum or minimum values is called a set of principal axes of the area $A$.

The corresponding moments of inertia $I_{x'}$ & $I_{y'}$ is called principal moments of inertia.

Using Trigonometry identity:

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad \cos 2\theta = 2 \cos^2 \theta - 1 \quad = \cos^2 \theta - \sin^2 \theta.$$\quad \text{in (2), (3), (4)}.

$$I'_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad \rightarrow (6)$$

$$I'_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad \rightarrow (7)$$

$$I'_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \quad \rightarrow (8)$$

Set the angle at which $I_{x'}$ is maximum or minimum be $\theta_p$. 


To calculate $\Theta_p$:

Take derivative of (6) with respect to $2\theta$ &
set it to zero:

$$\frac{\partial 2\theta}{\partial \theta} = \frac{2I_x y}{I_y - I_x}$$

You will get the same result if you do the same for $I_y$.

To determine principal axes and principal moments of inertia of an area:

1. Determine $I_x, I_y, I_z$
2. Determine $\Theta_p$
3. Calculate $I_{x'}$ & $I_{y'}$ using Equation (2) and
   Equation (3) or Eqn (6) & Eqn (7).
(a) Determine a set of principal axes and corresponding principal moments of inertia for the triangular area.

\( I_x = \frac{1}{12} (4)(3^3) = 9 \text{ m}^4 \)

\( I_y = \frac{1}{4} (4^3)(3) = 48 \text{ m}^4 \)

\( I_{xy} = \frac{1}{8} (4^2)(3^2) = 18 \text{ m}^4 \)

(b) Determine \( \Theta_p \)

\[
\tan \Theta_p = \frac{2I_{xy}}{I_y - I_x} = \frac{2(18)}{48 - 9} = 0.923
\]

\[\Theta_p = 21.4^\circ\]

(c) Now to calculate \( I_x \) and \( I_y \) using \( \Theta_p = 21.4^\circ \)

\[ I_x = \frac{I_x + I_y + (I_x - I_y) \cos 2\Theta - I_{xy} \sin 2\Theta}{2} \]

\[ = \frac{(9 + 48) + (9 - 48) \cos(2(21.4^\circ))}{2} - 18 \sin(2(21.4^\circ)) \]

\[ \approx 1.96 \text{ m}^4 \]
\[ I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \]

\[ = \left( \frac{9 + 48}{2} \right) - \left( \frac{9 - 48}{2} \right) \cos \left[ 2 \left( 21.4^\circ \right) \right] \]

\[ + 18 \sin \left[ 2 \left( 21.4^\circ \right) \right] = 55 \text{ m}^4 \]

Moina's Circle