General-Relativistic Gravitational Collapse

One of first detailed numerical calculations of the collapse of a spherically symmetric massive star was done by M.M. May and R.H. White, Phys. Rev. 141, 1232-1241 (1966).

![Graph showing $R(\mu,t)/R(\mu,0)$ versus $t\left[2MG/R_0^3(\mu_{\text{max}})^{1/2}\right]$ for various mass fractions during the collapse and bounce of 2.1, 21, and 210 $M_\odot$, $\gamma=5/3$ spheres. The initial conditions in each case were $R_e/R=6.2\times10^{-3}$, $\epsilon_0/\rho^2=3.84\times10^{-5}$ corresponding to a $K$ (Eq. 22) of 2.]

**Fig. 1.** $R(\mu,t)/R(\mu,0)$ versus $t\left[2MG/R_0^3(\mu_{\text{max}})^{1/2}\right]$ for various mass fractions during the collapse and bounce of 2.1, 21, and 210 $M_\odot$, $\gamma=5/3$ spheres. The initial conditions in each case were $R_e/R=6.2\times10^{-3}$, $\epsilon_0/\rho^2=3.84\times10^{-5}$ corresponding to a $K$ (Eq. 22) of 2.
Collapse Equations for an Ideal Fluid

The equations solved by May and White were derived in a very clear article by C.W. Misner and D.H. Sharp, Phys. Rev. 136, B571-B576 (1964)

The general-relativistic equations to be solved are

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \]

where the metric tensor \( g^{\mu\nu} \) is defined by the spherically symmetric form

\[ ds^2 = -e^{2\phi}dt^2 + e^\lambda dr^2 + R^2d\Omega^2, \quad d\omega^2 = d\theta^2 + \sin^2 \theta d\phi^2, \]

where the metric functions \( \phi(r,t), \lambda(r,t) \) and \( R(r,t) \) are independent of \( \Omega = (\theta, \varphi) \).

Ideal Fluid Stress-Energy Tensor

The initial state of the star is assumed to be a spherically symmetric ideal fluid distribution specified by four-velocity \( u^\mu \), internal energy density \( \epsilon \) and pressure \( p \) with stress energy tensor

\[ T_{\mu\nu} = (p + \epsilon)u^\mu u^\nu + pg^{\mu\nu}. \]

- In non-relativistic thermodynamics \( p(r,t) \) and \( \epsilon(r,t) \) are scalars. In general relativity stresses (pressure, shear) are tensor components, and energy is the time component of a four-vector. In this application
the quantities $p$ and $\epsilon$ are defined at each spacetime point $x^\mu$ to be the values of pressure and energy density in the rest frame of the fluid at that point: these local rest-frame quantities are scalar functions.

- The energy density $\epsilon$ is defined as the internal energy of the fluid per unit proper rest volume, i.e., the energy per unit volume of the fluid in the local rest frame.

- In non-relativistic mechanics mass is conserved. In general relativistic collapse baryon number is conserved. The scalar baryon number density $n(r, t)$ is defined as the number of baryons per unit proper rest volume, and the specific volume $v(r, t) = 1/n(r, t)$.

### Comoving Coordinate System and Metric Functions

The GR equations are form-invariant under differential local coordinate transformations. This can be used to define a coordinate system comoving with the fluid. In this reference system the fluid is instantaneously at rest with four-velocity

$$u^t = \frac{dt}{d\tau} = e^{-\phi}, \quad u^i = 0, \quad i = r, \theta, \varphi.$$

The equations of motion can be simplified by defining

$$U = D_t R \equiv u^\mu \frac{\partial R(r, t)}{\partial x^\mu} = e^{-\phi} \dot{R},$$

where $D_t$ is the comoving proper time derivative, so that $U d\theta$ gives the relative speed of two adjacent fluid element at the same radial coordinate $r$ and angular separation $d\theta$ as they move radially in or out. The
function \( \phi(r, t) \) in the metric is replaced by

\[
e^{2\phi} = \left( \frac{\dot{R}}{U} \right)^2.
\]

In the static case the mass \( m(r) \) within a radius \( r \) was used to replace the radial metric function

\[
g_{rr} = e^{\lambda(r)} = \left( 1 - \frac{2m(r)}{r} \right)^{-1}.
\]

The appropriate generalization to the time-dependent case is

\[
e^{\lambda(r,t)} = \left[ 1 + U^2 - \frac{2m(r,t)}{R} \right]^{-1} \left( \frac{\partial R}{\partial r} \right)^2,
\]

which reduces to the static case where \( R(r, t) = r \) and \( U = 0 \).

**Equation of State**

As in the static case, it is necessary to specify the relation between the pressure \( p(r, t) \) and energy density \( \epsilon(r, t) \) on the right hand side of the general relativistic equations of motion.

The simplest possible approximation is an ideal gas of baryons in thermal equilibrium at temperature \( T = 0 \) and the flow is adiabatic, i.e., the energy of a fluid element does not change as it moves radially in or out.
The internal energy density is determined by the baryon number density, given the adiabatic equation of state

\[ \epsilon = \epsilon(n) , \]

and the pressure is then determined by the thermodynamic relation

\[ p = n \left( \frac{\partial \epsilon}{\partial n} \right)_s - \epsilon . \]

where \( s \) is the specific entropy.

The specific internal energy \( \epsilon/n \) and pressure \( p \) of unit amount of fluid with specific volume \( v = 1/n \) containing a mole of baryons determine the specific Enthalpy

\[ h = \frac{\epsilon}{n} + pv = \frac{\epsilon + p}{n} = \left( \frac{\partial \epsilon}{\partial n} \right)_s . \]

Conservation Equations

The equations of fluid dynamics are based on various conservation laws. The stress-energy tensor is conserved

\[ T_{\mu\nu} ; \nu = 0 , \]

which relates the metric function \( \phi(r, t) \) to the specific enthalpy

\[ e^\phi = (-g_{00})^{1/2} = \frac{1}{h} . \]
The equation of continuity of baryon number

\[(n(r, t) u^\mu)_{;\mu} = 0\]

can be written in the form

\[\frac{4\pi R^2 n}{\sqrt{1 + U^2 - 2m/R}} \frac{\partial R}{\partial r} = \left(\frac{dA}{dr}\right)_{t=0},\]

where \(dA\) is the amount of matter in a spherical shell of thickness \(dr\) is independent of time.

**Einstein’s Equations**

Einstein’s equations can be simplified using the functions defined above. The derivation is done in detail by Misner and Sharp. There are three dynamical equations

\[D_t R = U,\]
\[D_t m = -4\pi R^2 pU,\]
\[D_t U = -\left[\frac{1 + U^2 - 2m/R}{\epsilon + p}\right] \left(\frac{\partial p}{\partial R}\right)_t - \frac{m + 4\pi R^3 p}{R^2},\]

and one kinematical equation

\[\left(\frac{\partial m}{\partial R}\right)_t = 4\pi R^2 \epsilon,\]

in addition to the equation of state and the two conservation equations.
Boundary Conditions and Initial Conditions

To solve these equations numerically, the initial values of the functions \( R(r, 0) \), \( m(r, 0) \) and \( U(r, 0) \) are specified arbitrarily.

A unique solution is obtained for this hyperbolic system boundary conditions at \( r = 0 \) and the surface of the star at \( r_s \) defined by

\[
  p = 0 \quad \text{at} \quad r = r_s = \text{constant},
\]

which determines the mass of the star

\[
  M = m(r_s, t)
\]

and the exterior Schwarzschild metric

\[
  ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2.
\]

The boundary conditions at the center of the star are

\[
  R(0, t) = m(0, t) = U(0, t) = 0.
\]

Finite Difference Equations to be Solved

May and White (Phys. Rev. 141, 1232-1241 (1966)) solved these equations numerically for \( M = 2.1 M_\odot \), \( 21 M_\odot \) and \( 210 M_\odot \) using an adiabatic equation of state \( P = \frac{2\rho\epsilon}{3} \) corresponding to an adiabatic index \( \gamma = 5/3 \). The initial conditions were taken to uniform inside the star.
Their equations were improved by K.A. van Riper, "General relativistic hydrodynamics and the adiabatic collapse of stellar cores", Astrophys. J. 232, 558-571 (1979) who gives explicit formulas convenient for coding in an appendix.

The following difference equations advance the model from time \( n \) to time \( n + 1 \). We begin by integrating the equation for \( e^\omega \) inward from the surface boundary condition:

\[
\left\{ e^\omega = \frac{1 - 2G\hat{m}/(c^2R)}{[1 + (U^{n-1/2}/c)^2 - 2G\hat{m}/(c^2R)^{1/2}]} \right\}^n, \quad (A1)
\]

\[
\{ w = \frac{1 + B + (P + Q)V[c^2(T+1/2)^2]}{\Gamma} \}^{1/2/2}, \quad (A2)
\]

\[
\{ e^{\omega-1/2} = e^{\omega-1} \exp \left[ \sum (P_{j-1/2} - P_{j-3/2} + Q_{j-1/2} - Q_{j-3/2}) V_{j-1/2}/(c^2) \right]\}^n, \quad (A3)
\]

\[
\{ e^{\omega-1} = e^{\omega-1} \exp \left[ \sum (P_{j-1/2} + Q_{j+1/2} - P_{j-1/2} + Q_{j-1/2}) V_{j-1/2}/(c^2) \right]\}^n, \quad (A4)
\]

\[
\{ e^{\omega-1} = e^{\omega-1} \exp \left[ \sum (P_{j-1/2} + Q_{j+1/2} - P_{j-1/2} + Q_{j-1/2}) V_{j-1/2}/(c^2) \right]\}^n, \quad (A5)
\]

\[
\{ \tilde{m}_{j+1} = \tilde{m}_j + (1 + B_{j+1/2})[(\Gamma_j + \Gamma_{j+1})d\tilde{m}_{j+1/2}]_{j+1/2} \}, \quad (A6)
\]

\[
U_j^{n+1/2} = U_j^{n-1/2} + (\Delta t e^\omega \sum (4\pi G\hat{m})(\Gamma_j + \Gamma_{j+1})d\tilde{m}_{j+1/2})^{n+1/2} - G\hat{m}_j (R_j z^2 - 4\pi G\hat{m})(\Gamma_j + \Gamma_{j+1})d\tilde{m}_{j+1/2})^{n+1/2}, \quad (A7)
\]

\[
U_j^{n+1/2} = U_j^{n-1/2} - \left\{ \Delta t e^\omega \sum (16\pi G\hat{m})(\Gamma_j + \Gamma_{j+1})d\tilde{m}_{j+1/2})^{n+1/2} - G\hat{m}_j (R_j z^2 - 4\pi G\hat{m})(\Gamma_j + \Gamma_{j+1})d\tilde{m}_{j+1/2})^{n+1/2}, \quad (A8)
\]

\[
R_j^{n+1} = R_j^{n} + \Delta t e^\omega \sum (4\pi G\hat{m})(\Gamma_j + \Gamma_{j+1})d\tilde{m}_{j+1/2})^{n+1/2}, \quad (A9)
\]

\[
\{ e^{\omega+1} = \frac{1 + (U^{n+1/2}/c)^2 - 2G\hat{m}/(c^2R)^{1/2})}{\Gamma} \}^{1/2/2}, \quad (A10)
\]

\[
\{ V_j^{n+1/2} = (8\pi/3)[(R_j z^2 - (R_j z^2)]^{n+1/2} - (R_j z^2)]^{n+1/2}\}^{n+1/2}, \quad (A11)
\]

\[
DU = U_j^{n+1/2} - U_j^{n-1/2} \}, \quad (A12)
\]

\[
Q_j^{n+1} = 0 \quad \text{if } DU \geq 0
\]

\[
= DU^2/2V_j^{n+1/2} \quad \text{if } DU < 0 \quad (j = 1, J - 1), \quad (A13)
\]

The adiabatic energy equation, \( dE = -PdV \), involves only local thermodynamic quantities and is not changed by GR. With the artificial viscosity, we want to solve

\[
\{ E^{n+1} - E^n = -\frac{1}{4}(P^n + Q^n + P^{n+1} + Q^{n+1})(V^{n+1} - V^n) \}^{n+1/2} \}, \quad (A14)
\]