Equilibrium of White Dwarfs and Neutron Stars

These are stars that have exhausted their nuclear fuel. They can be modeled as balls of ideal Fermi gases at temperature $T = 0$ with fermion degeneracy (Pauli exclusion) pressure balancing gravity. In white dwarfs the pressure is generated by degenerate electrons, and in neutron stars by degenerate neutrons.

A first order of magnitude estimate was suggested by Landau. Consider $N$ fermions in a star of radius $R$. The number density $n \sim N/R^3$. A fermion confined to a volume $\sim 1/n$ has momentum $\sim \hbar n^{1/3}$ by Heisenberg’s uncertainty principle. The energy of a fermion is comparable to the Fermi energy

$$E_F \sim \hbar n^{1/3} c \sim \frac{\hbar c N^{1/3}}{R}$$

The gravitational potential energy of a fermion is

$$E_G \sim -\frac{GMm_B}{R} \quad M = Nm_B$$

where $m_B$ is the baryon mass (per electron for white dwarfs or the neutron mass for neutron stars).

In static equilibrium at $T = 0$, the total energy is a minimum

$$E = E_F + E_G = \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

Note that both terms scale like $1/R$ so the equilibrium is determined by the sign of $E$. 
If $N$ is small enough $E > 0$ and is lowered by increasing $R$. This causes $E_F$ to decrease. The fermions will become non-relativistic

$$E_F \sim \frac{p_F^2}{2m} \sim \frac{\hbar^2 N^{2/3}}{2mR^2}$$

and there will be a stable minimum at

$$\frac{dE}{dR} \sim -\frac{\hbar^2 N^{2/3}}{R^3} + \frac{GNm_B^2}{R^2} = 0$$

If $N$ is large enough the total energy can become negative $E < 0$ and decreases without bound as gravity causes the star to collapse.

These two limiting cases imply that there is a maximum baryon number for equilibrium given by

$$E = 0, \quad N_{\text{max}} \sim \left( \frac{\hbar c}{Gm_B^2} \right)^{3/2} \sim 2 \times 10^{57}, \quad M_{\text{max}} \sim N_{\text{max}}m_B \sim 1.5 \, M_\odot.$$  

If $N = N_{\text{max}}$ and the fermions are relativistic $E_F \geq mc^2$

$$R \leq \frac{\hbar}{mc} \left( \frac{\hbar c}{Gm_B^2} \right)^{1/2} \sim \begin{cases} 5 \times 10^{3} \, \text{km}, & m = m_e \quad \text{(white dwarfs)} \\ 3 \, \text{km}, & m = m_n \quad \text{(neutron stars)} \end{cases}$$

This is essentially the physics of the Chandrasekhar and Tolman-Oppenheimer-Volkoff limits.
White Dwarfs

The theory of white dwarfs was first developed by S. Chandrasekhar \( \text{Mon. Not. Roy. Astron. Soc. 95,207-225 (1935)} \) based on modeling the electrons as a \( T = 0 \) ideal Fermi gas

\[
E_F = \sqrt{p_F^2 c^2 + m_e^2 c^4} \quad , \quad n_e = \frac{2}{\hbar^3} \int_0^{p_F} dp \ 4\pi p^2 = \frac{8\pi}{3\hbar^3} p_F^3 = \frac{1}{3\pi^2 \lambda_e^3} x^3 \quad , \quad x \equiv \frac{p_F}{m_e c}
\]

where \( \lambda_e = \hbar/(m_e c) \) is the Compton wavelength of the electron. The pressure and energy density of the
gas are given by

\[
P_e = \frac{8\pi m_e^4 c^5}{3\hbar^3} \int_0^x dx \frac{x^4}{\sqrt{1 + x^2}}, \quad \varepsilon_e = \frac{8\pi m_e^4 c^5}{3\hbar^3} \int_0^x dx \frac{x^2}{\sqrt{1 + x^2}}.
\]

More realistic results take into account Coulomb corrections, and various nuclear effects.

Neutron Stars

The physics of white dwarfs is essentially determined by the properties of a single elementary particle, the electron. The physics of neutron stars is determined by neutrons, protons and many of the stable atomic nuclei, and the strong and weak interactions play an essential role in addition to degeneracy pressure and Coulomb forces.

Neutron star models suggest a structure similar to a planet, with a solid crust, an intermediate mantle, and a liquid core:

From CERN Courier (March 2013)
- There is an outer envelope which is probably a lattice of Fe ions with an electron gas.
- The outer crust is a Coulomb lattice of electrons and neutron-rich nuclei.
- The inner crust consists of neutrons, nuclear clusters and electrons.
- A mantle separates the solid crust from the liquid core.
- The central core is likely a liquid of neutrons, protons and electrons.
- In a sufficiently massive star a quark-gluon liquid may develop at the center of the core.
Figure 4: Schematic picture of the ground state structure of neutron stars along the density axis. Note that the main part of this figure represents the solid crust since it covers about 14 orders of magnitude in densities.

From Living Rev. Relativity 11 (2008), 10
Tolman-Oppenheimer-Volkoff Equilibrium Equations

Consider a spherical star described by a perfect fluid in equilibrium with its gravitational field. The Tolman-Oppenheimer-Volkoff equations using the notation of Baym, Pethick and Sutherland Astrophys. J. 170, 299-317 (1971) is

$$\frac{dP(r)}{dr} = -\frac{G}{r^2} \left[ \rho(r) + \frac{P(r)}{c^2} \right] \left[ m(r) + \frac{4\pi r^3 P(r)}{c^2} \right] \left[ 1 - \frac{2Gm(r)}{rc^2} \right]^{-1}$$

where the $P(r)$ and $\rho(r)$ are the pressure and density at radial coordinate $r$, and

$$m(r) = \int_0^r d^3r' \rho(r')$$

is mass contained within a sphere of radius $r$. The radius of the star is the radial coordinate value at which the pressure $P(R) = 0$, and total mass of the star is

$$M = m(R) = \int d^3r \rho(r) .$$

The number of baryons within radius $r$ is

$$b(r) = \int_0^r d^3r' \frac{n_b(r')}{\sqrt{1 - 2Gm(r')/(r'c^2)}}$$

where $n_b(r)$ is the baryon number density, and the total number of baryons is

$$B = b(R) .$$
To determine the equilibrium configuration of the star, the TOV equation must be integrated from $r = 0$ to $r = R$ and matched onto the external Schwarzschild solution.

Note that there are two unknown functions in the equation, the pressure $P(r)$ and the density $\rho(r)$. To close the system, an equation of state (EOS) for nuclear matter must be specified.

Baym, Pethick and Sutherland Astrophys. J. 170, 299-317 (1971) developed an EOS applicable to nuclear matter and obtained the results shown in the Figure:
Fig. 1.—Mass versus central density for zero-temperature nonrotating stars in nuclear equilibrium. Stars to the left of the maximum (the Chandrasekhar limit) at $\rho_c = 1.4 \times 10^9$ g cm$^{-3}$ are stable white dwarfs, while stars to the right of the minimum at $1.55 \times 10^{14}$ g cm$^{-3}$ are neutron stars. The lower dashed extension of the curve was constructed from Pandharipande's hyperonic equation of state $C$, and the upper dashed extension from Pandharipande's equation of state for pure neutrons. Neutron stars beyond the maximum are unstable.
The general relativistic adiabatic index (ratio of specific heats) is

$$\Gamma = \frac{n_b \frac{\partial P}{\partial n_b}}{P \frac{\partial n_b}{P}} = \frac{\rho + P/c^2}{P} \frac{\partial P}{\partial \rho}.$$