Basic Formulas for Cross Sections and Decay Rates

The \textbf{Scattering cross section} is defined by Peskin-Schroeder on page 100 by the formula

\[ \sigma \equiv \frac{\text{Number of scattering events of a given type}}{\rho_A \ell_A \rho_B \ell_B A}. \]

where \( A \) is the overlapping cross sectional area of the two beams. The differential cross section is

\[ d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left( \prod_{f} \frac{d^3p_f}{(2\pi)^3 2E_f} \right) \left| \mathcal{M}(p_A, p_B \rightarrow \{p_f\}) \right|^2 (2\pi)^4 \delta(4) \left( p_A + p_B - \sum p_f \right), \]

Eq. (4.79). The matrix element \( \mathcal{M} \) is simply the sum of Feynman diagrams to which the Feynman rules are applied. The formula for the decay of an unstable particle is given in Eq. (4.86)

\[ d\Gamma = \frac{1}{2m_A} \left( \prod_{f} \frac{d^3p_f}{(2\pi)^3 2E_f} \right) \left| \mathcal{M}(m_A \rightarrow \{p_f\}) \right|^2 (2\pi)^4 \delta(4) \left( p_A - \sum p_f \right). \]
FeynCalc: Tools for Field Theory Calculations

Download and install FeynCalc from the website http://www.feyncalc.org/ or directly from a Mathematica Notebook by invoking the Mathematica Import function:

```
Import["http://www.feyncalc.org/install.m"]
```

The second In definition statement Imports and evaluates the Mathematica code Bhabha.m from Dr. Romão’s web site http://porthos.ist.utl.pt/CTQFT/node3.html.
Bhabha Scattering Cross Section using FeynCalc

J.C. Romão, *Computational Techniques in Quantum Field Theory* has a good tutorial and example codes for relativistic perturbation theory calculations using FeynCalc and other computer programs.

Bhabha scattering is the elastic electron-positron scattering process

\[ e^{-}(p_1) + e^{+}(p_2) \rightarrow e^{-}(p_3) + e^{+}(p_4). \]

The matrix element for the process is a sum of \(t\)-channel and \(s\)-channel contributions

\[ \mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \frac{e^2}{t} \bar{u}(p_3)\gamma^{\mu}u(p_1)\bar{v}(p_2)\gamma_{\mu}v(p_4) - \frac{e^2}{s} \bar{v}(p_2)\gamma^{\mu}u(p_1)\bar{u}(p_3)\gamma_{\mu}v(p_4), \]

where the **Mandelstam variables** \(s, t, u\) are defined by

\[ t = (p_1 - p_3)^2, \quad s = (p_1 + p_2), \quad u = (p_1 - p_4)^2, \quad s + t + u = 4m_e^2. \]
Feynman Rules for QED

From Peskin-Schroeder pages 118 and 123:

We denote scalar particles by dashed lines, and fermions by solid lines. The $S$-matrix element could then be obtained directly from the following momentum-space Feynman rules.

1. Propagators:

\[
\phi(x)\phi(y) = \frac{i}{q^2 - m^2 + i\epsilon}
\]

\[
\bar{\psi}(x)\bar{\psi}(y) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}
\]

2. Vertices:

\[
\begin{array}{c}
\hline
\end{array}
\]

\[
= -ig
\]

3. External leg contractions:

\[
\phi(q) = \begin{array}{c}
\hline
\end{array} = 1
\]

\[
\langle q | \phi = \begin{array}{c}
\hline
\end{array} = 1
\]

\[
\begin{array}{c}
\hline
\end{array} = u^\alpha(p)
\]

\[
\begin{array}{c}
\hline
\end{array} = \bar{u}^\alpha(p)
\]

\[
\begin{array}{c}
\hline
\end{array} = \bar{v}^\alpha(k)
\]

\[
\begin{array}{c}
\hline
\end{array} = v^\alpha(k)
\]

4. Impose momentum conservation at each vertex.

5. Integrate over each undetermined loop momentum.

6. Figure out the overall sign of the diagram.
Mathematica Code Bhabha.m

Mathematica [Input Syntax] defines the meaning of Mathematica programs.

The code begins with a Mathematica [Bracketed comment].

(* Program to Calculate the Traces of Eq. 4.43 de TC

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*)

It is generally a good idea to clear Mathematica’s memory of all previous calculations to avoid conflicting definitions and other strange bugs. This is done by invoking [Remove] on all symbols in the global [Standard Namespace]

Remove["Global`*""]

The next statements invoke [FeynCalc functions] to Define Mathematica Functions to generate

\[ \gamma^\mu, \gamma^5, \not{p} = \gamma \cdot p = \gamma^\mu p_\mu, \ g^{\mu\nu}, \ p^\mu, \ \epsilon^{\mu\nu\rho\sigma}, \ 1_{n \times n}, \ p \cdot q, \ \not{p} + m, \ \gamma_{R,L} = \frac{1 \pm \gamma^5}{2}. \]

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dm[\mu_]:={\text{DiracMatrix}}[\mu] 

dm[5]:={\text{DiracMatrix}}[5] 

ds[p_]:={\text{DiracSlash}}[p] 

mt[\mu_,\nu_]:={\text{MetricTensor}}[\mu,\nu] 

fv[p_,\mu_]:={\text{FourVector}}[p,\mu] 

\epsilon[\alpha,\beta,\gamma,\delta] := {\text{LeviCivita}}[\alpha,\beta,\gamma,\delta] 

id[n_]:={\text{IdentityMatrix}}[n] 

sp[p_,q_]:={\text{ScalarProduct}}[p,q] 

li[\mu_]:={\text{LorentzIndex}}[\mu] 

prop[p_,m_]:={\text{ds}}[p]+m 

PR = (1 + dm[5])/2 

PL = (1 - dm[5])/2 

Next, define and set variables Line1, \ldots, Line5 to represent products of Dirac matrices 

Line1 := ds[p3] \cdot dm[\mu] \cdot ds[p1] \cdot dm[\nu] 

Line2 := ds[p2] \cdot dm[\mu] \cdot ds[p4] \cdot dm[\nu] 

Line3 := ds[p2] \cdot dm[\mu] \cdot ds[p1] \cdot dm[\nu] 

Line4 := ds[p3] \cdot dm[\mu] \cdot ds[p4] \cdot dm[\nu]

Next invoke the FeynCalc functions

- \textbf{Tr} to perform the Trace of Dirac matrix products,
- \textbf{Contract} to contract repeated Lorentz indices

and the Mathematica \textbf{Simplify} function to simplify the resulting expressions:

\begin{align*}
\text{ans1} &= \text{Simplify}[\text{Contract}[\text{Tr[Line1]} \ \text{Tr[Line2]}]] \\
\text{ans2} &= \text{Simplify}[\text{Contract}[\text{Tr[Line3]} \ \text{Tr[Line4]}]] \\
\text{ans3} &= \text{Simplify}[\text{Contract}[\text{Tr[Line5]}]] \\
\text{ans} &= \frac{\text{ans1}}{4}t^2 + \frac{\text{ans2}}{4}s^2 - 2\frac{\text{ans3}}{4}s/t
\end{align*}

Define a \textbf{List of Replace Rules} to make substitutions in an expressions

\begin{verbatim}
dot={sp[p1,p2]->s/2,sp[p3,p4]->s/2,sp[p1,p3]->-t/2,sp[p2,p4]->-t/2, sp[p1,p4]->(s+t)/2,sp[p2,p3]->(s+t)/2}
\end{verbatim}
ReplaceAll

res = e^4 ans /. dot

and Print the answer using \LaTeX formatting:

Print["\!\!(\!*FractionBox[\!(\!(1\!), \!(4\!)\!)\!) \!\!(\!*UnderscriptBox[\!(\!(\![\text{Sum}]\!,\!)\!), \!(\!(\text{spins}\!)\!)\!)\!]M\!\!(\!*SuperscriptBox[\!(\!(1\!), \!(2\!)\!)\!] = ",res]

(*********************** Fim do Programa ***********************

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