Multidimensional Gaussian Integrals

A one-dimensional Gaussian integral can be done exactly with a shift of integration variable

\[
\int_{-\infty}^{\infty} dx \, e^{-\frac{a}{2}x^2 + bx} = \left[ \int_{-\infty}^{\infty} dx \, e^{-\frac{a}{2}(x-b)^2} \right] \frac{b^2}{2a} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}
\]

Gaussian Integral in One Complex Variable

The one-dimensional real Gaussian integral can be generalized to a complex variable \( z = x + iy \) where \( x \) and \( y \) are independent real variables

\[
\int d(\bar{z}, z) e^{-\bar{w}z\bar{z}} \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, e^{-w(x^2+y^2)} = \frac{\pi}{w}, \quad \Re w \equiv \frac{\bar{w} + w}{2} > 0,
\]

where \( \bar{z} = x - iy \). The integrations converge if the real part \( \Re w \) of \( w \) is positive definite.

This can be generalized to a complex Gaussian shifted two complex different numbers \( u, v \)

\[
\int d(\bar{z}, z) e^{-\bar{w}z\bar{z} + \bar{u}z - \bar{v}v} = \left( \frac{\pi}{w} \right) e^{\frac{\bar{w}v}{w}}.
\]
Real Multidimensional Gaussian

If \( \mathbf{v} \) is a real vector variable with \( N \) components, \( \mathbf{A} \) is a real symmetric \( N \times N \) matrix, and \( \mathbf{j} \) is a real \( N \)-dimensional vector,

\[
\int d^N \mathbf{v} \ e^{-\frac{1}{2} \mathbf{v}^T \mathbf{A} \mathbf{v} + \mathbf{j}^T \mathbf{v}} = (2\pi)^{N/2} \det \mathbf{A}^{-1/2} \exp \left[ \frac{1}{2} \mathbf{j}^T \mathbf{j} \right],
\]

which can be proved by diagonalizing \( \mathbf{A} \) by orthogonal transformation

\[
\mathbf{A} = \mathbf{O}^T \begin{pmatrix}
  d_1 & 0 & 0 & \cdots & 0 \\
  0 & d_2 & 0 & \cdots & 0 \\
  \vdots & & & \ddots & \\
  0 & 0 & 0 & \cdots & d_N
\end{pmatrix} \mathbf{O}, \quad d_i > 0, \quad i = 1, \cdots, N, \quad \det \mathbf{A} = \prod_{j=1}^{N} d_j.
\]

Complex Multidimensional Gaussian

Altland-Simons give the generalization to a complex \( N \) component vector \( \mathbf{v} \neq \mathbf{v}^\dagger \) and non-Hermitian matrix

\[
\mathbf{A} = \frac{\mathbf{A} + \mathbf{A}^\dagger}{2} + \frac{\mathbf{A} - \mathbf{A}^\dagger}{2}.
\]

If the Hermitian component of \( \mathbf{A} \) is a positive definite matrix with real eigenvalues \( d_i > 0 \), then

\[
\int d(\mathbf{v}^\dagger, \mathbf{v}) \ \exp \left[ -\mathbf{v}^\dagger \mathbf{A} \mathbf{v} + \mathbf{w}^\dagger \mathbf{v} + \mathbf{v}^\dagger \mathbf{w}' \right] = \pi^N \ \det \mathbf{A}^{-1} \exp \left[ \mathbf{w}^\dagger \mathbf{A}^{-1} \mathbf{w}' \right].
\]
Lagrangian Path Integral Formulation of Quantum Mechanics

The Hamiltonian formulation of the path integral for the propagator

\[
\langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle \simeq \int \prod_{n=1}^{N-1} dq_n \prod_{n=1}^{N} dp_n \exp \left[ -\frac{i\Delta t}{\hbar} \sum_{n=0}^{N-1} \left( V(q_n) + T(p_{n+1}) - \frac{p_{n+1}(q_{n+1} - q_n)}{\Delta t} \right) \right]
\]

follows from dividing the time interval \( t \) into \( N \) equal steps of size \( \Delta t \) and factorizing the time evolution operator into a product of kinetic and potential energy parts.

The integrals over the time step momenta \( p_n \)

\[
\prod_{n=1}^{N} \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp \left[ -\frac{i\Delta t}{\hbar} \sum_{n=0}^{N-1} \left( \frac{p_n^2}{2m} - \frac{p_{n+1}(q_{n+1} - q_n)}{\Delta t} \right) \right]
\]

can be done exactly because the integrand is the exponential of a quadratic form.

Wick Rotation to Euclidean Space

The quantum time evolution operator is not well defined as a product of Gaussian integrals because the Gaussian factors oscillate with increasing frequency as \( p_n, q_n \to \pm \infty \).

They can be defined by making a Wick rotation to imaginary time \( t \to \tau = -it \), or equivalently, by rotating
the contour of the $p_n$ integration counterclockwise by $90^\circ$ in the complex plane, see Peskin-Schroeder pages 192-193.

This procedure was introduced by G.C. Wick, *Properties of Bethe-Salpeter Wave Functions*, Phys. Rev. 96, 1124–1134 (1954), to study the Bethe-Salpeter equation wave function

$$
\phi_1(p, p_0) = \frac{1}{(2\pi)^4} \int d^3x \ e^{-ip \cdot x} \int_0^{+\infty} dt \ e^{ip_0 t} \chi(x) = \frac{1}{2\pi i} \int_{\omega_{\text{min}}}^{+\infty} d\omega \ \frac{f(p, \omega)}{\omega - p_0 - i\epsilon}
$$
for the relativistic bound state of a two-body system. Note the use of the Feynman $i\epsilon$ prescription to specify causal Green function boundary conditions.

**Lagrangian Path Integral**

Using the Wick rotation method to perform the $p_n$ integrations for $n = 1, \cdots, N$ transforms the propagator

$$\langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle = \int \prod_{n=1}^{N-1} dq_n \exp \left[ -\frac{i\Delta t}{\hbar} \sum_{n=0}^{N-1} \left\{ V(q_n) - \frac{1}{2} \left( \frac{q_{n+1} - q_n}{\Delta t} \right)^2 \right\} \right].$$

In the limit $\Delta t \to 0$

$$V(q_n) - \frac{1}{2} \left( \frac{q_{n+1} - q_n}{\Delta t} \right)^2 \to V(q) - \frac{1}{2} q^2 = -L(q, \dot{q}), \quad \Delta t \sum_{n=0}^{N-1} \to \int_0^t dt',$$

and the expression for the propagator becomes

$$\langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle = \int Dq \exp \left[ \frac{i}{\hbar} \int_0^t dt' L(q(t'), \dot{q}(t')) \right] = \int Dq \exp \left[ \frac{i}{\hbar} S[q] \right],$$

where $S[q]$ is the classical action functional for the trajectory $q(t')$ from $q(0) = q_i$ to $q(t) = q_f$. 
Path Integral as a Generator for Green Functions

Consider the real multidimensional Gaussian integral
\[ \int d^N v \, e^{-\frac{1}{2}v^T A v + j^T v} = (2\pi)^{N/2} \det A^{-1/2} \exp \left[ \frac{1}{2} j^T j \right], \]
take the double derivative of left- and right-hand sides and set the external source vector \( j \) to zero
\[ \left( \frac{\partial}{\partial j_m} \frac{\partial}{\partial j_n} \right) \left( \int d^N v \, e^{-\frac{1}{2}v^T A v + j^T v} = (2\pi)^{N/2} \det A^{-1/2} \exp \left[ \frac{1}{2} j^T j \right] \right) \bigg|_{j=0}, \]
gives the identity
\[ \int d^N v \, \exp \left[ -\frac{1}{2} v^T A v \right] v_m v_n = (2\pi)^{N/2} \det A^{-1/2} A^{-1}_{mn}, \]
which can be viewed as an average with respect to a normalized Gaussian probability distribution
\[ \langle v_m v_n \rangle = A^{-1}_{mn} \quad \text{where} \quad \langle \cdots \rangle \equiv (2\pi)^{-N/2} \det A^{+1/2} \int d^N v \, e^{-\frac{1}{2}v^T A v} (\cdots). \]

A 2\( n \)-fold differential gives the average
\[ \langle v_{i_1} v_{i_2} \cdots v_{i_{2n}} \rangle = \sum_{\text{all pairings}} A_{i_{k_1} i_{k_2}}^{-1} \cdots A_{i_{k_{2n-1}} i_{k_{2n}}}^{-1}. \]

The phase all pairings should remind you of the instruction \textbf{make all possible contractions} in Wick’s theorem! This will be the mathematical basis for deriving the Feynman rules in the path integral formalism.
Gaussian Functional Integration

This formulation for an $N$-component vector can be generalized to a functional integral over a real function $v(x)$, which can be viewed as a vector with a continuously infinite number of components

$$\int Dv(x) \exp \left[ -\frac{1}{2} \int dx \, dx' \, v(x) A(x, x') v(x') + \int dx \, j(x) v(x) \right]$$

$$\propto (\det A)^{-1/2} \exp \left[ \frac{1}{2} \int dx \, dx' \, j(x) A^{-1}(x, x') j(x') \right],$$

where $A(x, x')$ is a kernel with inverse $A^{-1}$ defined by

$$\int dx'' \, A(x, x'') A^{-1}(x'', x') = \delta(x - x').$$

The generalization of the $2n$-point Gaussian average is

$$\langle v_{x_1} v_{x_2} \cdots v_{x_{2n}} \rangle = \sum_{\text{all pairings}} A^{-1}(x_{k_1}, x_{k_2}) \cdots A^{-1}(x_{k_{2n-1}}, x_{k_{2n}}).$$