Spacetime Approach to Nonrelativistic Quantum Mechanics

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Space-Time Approach to Non-Relativistic Quantum Mechanics

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Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching $x, t$ from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schrödinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

Rev. Mod. Phys. 20, 367–387 (1948) “A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time”.

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Altland-Simons Chapter 3 has many applications: free particle, quantum well, double well, tunneling and instantons, decay of false vacuum (cosmological phase transitions), dissipative quantum tunneling, spin. Chapter 4 Section 4.1 Construction of the many-body path integral: Coherent states and Grassmann variables for fermions. Page 96: Path integrals (1) use classical solutions as central ingredients and the classical limit is constantly visible, (2) allow efficient formulation of non-perturbative approaches, and (3) are useful in many-body problems with macroscopically large collective variables.

Peskin-Schroeder Chapter 9: Section 9.1 Path Integrals in Quantum Mechanics, Section 9.5 Anticommuting (Grassmann) numbers, and Problem 9.2 Quantum statistical mechanics. Page 275: Path integrals (1) allow easy derivation of Feynman rules, (2) use the Lagrangian formalism and preserve all symmetries of a theory, (3) relate quantum field theory and statistical mechanics. Feynman said “every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics”.

Figure 3.1 (a) Visualization of a set of phase space points contributing to the discrete time configuration integral (3.5). (b) In the continuum limit, the set of points becomes a smooth curve.

Figure 9.2. We define the path integral by dividing the time interval into small slices of duration $\epsilon$, then integrating over the coordinate $x_k$ of each slice.
Hamiltonian Formulation of the Path Integral

The path integral represents the amplitude for the system to evolve from a position eigenstate with eigenvalue $q_i$ at initial time $t_i$ to a position eigenstate $q_f$ at $t_f$:

$$\langle q_f | e^{-i \hat{H} t / \hbar} | q_i \rangle = \int_{q(0)=q_i}^{q(t)=q_f} Dx \exp \left[ \frac{i}{\hbar} \int_0^t dt' \{ p \dot{q} - H(p, q) \} \right]$$

The symbol $D$ represents functional integration. This is the generalization of multi-dimensional integration over the components of a vector variable to a vector with a continuous infinity of components. It is defined mathematically by a limiting process. Functional calculus, or Functional analysis, is the generalization of multivariable calculus.

From particles to fields

Table 1.1 Summary of basic definitions of discrete and continuum calculus.

<table>
<thead>
<tr>
<th>Entity</th>
<th>Discrete</th>
<th>Continuum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argument</td>
<td>vector $f$</td>
<td>function $f$</td>
</tr>
<tr>
<td>Function(al)</td>
<td>multidimensional function $F(f)$</td>
<td>functional $F[f]$</td>
</tr>
<tr>
<td>Differential</td>
<td>$dF_x(g)$</td>
<td>$DF_x[g]$</td>
</tr>
<tr>
<td>Cartesian basis</td>
<td>$e_n$</td>
<td>$\delta_x$</td>
</tr>
<tr>
<td>Scalar product $\langle \cdot, \cdot \rangle$</td>
<td>$\sum_n f_n g_n$</td>
<td>$\int dx f(x) g(x)$</td>
</tr>
<tr>
<td>&quot;Partial derivative&quot;</td>
<td>$\partial_{f_n} F(f)$</td>
<td>$\frac{\delta F[f]}{\delta f(x)}$</td>
</tr>
</tbody>
</table>
Definition of the Functional Limit

Divide the evolution time interval into \( N \gg 1 \) steps and approximate

\[
\exp \left( -\frac{i\hat{H}t}{\hbar} \right) = \left[ \exp \left( -\frac{i\hat{H}\Delta t}{\hbar} \right) \right]^N , \quad \Delta t = \frac{t}{N}
\]

In general \( \hat{H} = \hat{T} + \hat{V} \) and the kinetic and potential energy operators do not commute with one another.

To evaluate the exponential factorize

\[
e^{-i\hat{H}\Delta t/\hbar} = e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} + \mathcal{O}(\Delta t)^2
\]

Approximate the propagator

\[
\langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle \approx \langle q_f | \left[ e^{-i\hat{H}\Delta t/\hbar} \right]^N | q_i \rangle \approx \langle q_f \rangle \wedge e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} \wedge \cdots \wedge e^{-i\hat{T}\Delta t/\hbar} e^{-i\hat{V}\Delta t/\hbar} | q_i \rangle
\]

and use the completeness identities for position and momentum eigenstates

\[
1_q 1_p = \int dq \langle q | \int dp |p\rangle \langle p | q \rangle
\]

at each timestep \( t_n \) at the locations indicated by \( \wedge \).

Using the normalized free particle eigenfunctions

\[
\langle q|q' \rangle = \delta(q-q') , \quad \langle p|p' \rangle = 2\pi \delta(p-p') , \quad \langle q|p \rangle = \langle p|q \rangle^* = \frac{e^{ipq/\hbar}}{(2\pi\hbar)^{1/2}}
\]
the propagator

\[ \langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle \sim \int \prod_{n=1}^{N-1} dq_n \prod_{n=1}^{N} dp_n \frac{2\pi\hbar}{\Delta t} \exp \left[ -\frac{i\Delta t}{\hbar} \sum_{n=0}^{N-1} \left( V(q_n) + T(p_{n+1}) - \frac{p_{n+1}(q_{n+1} - q_n)}{\Delta t} \right) \right] \]

Note that the end points \( q_{r,i} \) are fixed, and there is an additional \( p_N \) integral at the first wedge insertion with \( q_N = q_f \) fixed.

The limiting process damps contributions from paths for which

\[ |p_{n+1}(q_{n+1} - q_n)| > O(\hbar) \]

and the integrand oscillates infinitely rapidly as

\[ \lim_{\Delta t \to 0} N = \lim_{\Delta t \to 0} \frac{t}{\Delta t} \to \infty \]

The \textit{sum over paths} is effectively a sum over smooth paths.

To make this mathematically rigorous, the multiple integral can be analytically continued to imaginary time. The exponent is now real and the damping is exponential. The path integral in imaginary time is the \textit{Wiener integral} encountered in Brownian motion and the partition function of classical statistical mechanics.