Discrete Spacetime Symmetries and Charge Conjugation

Classical Electrodynamics and Quantum Electrodynamics are invariant under continuous Poincaré transformations $\text{SO}(1,3)$. Noether’s theorem implies that energy, momentum and angular momentum are conserved. These theories are also invariant under the discrete transformations of spacetime $x^{\mu}$ generated by the 4-element discrete group including space reflection $P$ and time reversal $T$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad PT = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In addition to these discrete spacetime symmetries, the equations of electrodynamics apply equally to electrons and positrons, and to particles and their antiparticles in general. This is referred to as charge conjugation $C$. In classical electrodynamics it corresponds to reversing the sign of electric charge. In quantum field theory it corresponds to interchanging the ladder operators for particles and antiparticles.

Noether’s theorem applies continuous symmetries. There are no conserved currents corresponding to discrete symmetries. Invariance under discrete symmetries of the Hamiltonian and physical eigenstates of a quantum system imply selection rules for transitions. For example, transitions between states of
opposite parity are forbidden, and the number of particles minus the number of antiparticles cannot change in any reaction.

**Violation of Parity and Charge-Parity in Weak Interactions**

The interaction of the electromagnetic current with photons

\[ \mathcal{L}_{\text{int}, \text{QED}} = -e \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) \]

is invariant under \( P \). The analogous interaction of a charged \( W \) boson with the charged electron-neutrino current responsible for nuclear \( \beta \)-decay [Peskin-Schroeder Eq. (20.79)]

\[ \mathcal{L}_{\text{int}, W} = \frac{g}{\sqrt{2}} \bar{\psi}_e(x) \gamma^\mu \left( \frac{1 - \gamma^5}{2} \right) \psi_e^-(x) W^-_\mu(x) \]

is not invariant under \( P \) because it mixes vector and pseudovector bilinears that transform with opposite signs under \( P \). The effect was observed by C.S. Wu et al., *Experimental Test of Parity Conservation in Beta Decay*, Phys. Rev. 105, 1413–1415 (1957)

The weak interaction is also not invariant under charge conjugation \( C \) because massless electron-neutrinos \( \nu_e \) are left-handed with helicity \( h = -1/2 \) but electron-antineutrinos \( \bar{\nu}_e \) are right-handed with \( h = +1/2 \). The interaction is invariant under the combined symmetry transformation \( CP \) because \( P \) flips helicity.

the branching ratio

$$|\eta_{+-}| = \frac{|A(K^0_L \rightarrow \pi^+ \pi^-)|}{|A(K^0_S \rightarrow \pi^+ \pi^-)|} = (2.232 \pm 0.011) \times 10^{-3} , \quad K^0_L = \frac{K^0 + \bar{K}^0}{\sqrt{2}} , \quad K^0_S = \frac{K^0 - \bar{K}^0}{\sqrt{2}} ,$$

which should vanish if $CP$ is conserved because $L$ (long) and $S$ (short) mixtures of $K^0$ and its antiparticle $\bar{K}^0$ transform oppositely under $CP$, see Hyperphysics for a simple explanation.

$CP$ violation appears be necessary to account for the matter-antimatter asymmetry in the universe, see Wikipedia Baryogenesis.

For a summary of the current status of $CP$ and $T$ symmetries see Tests of Conservation Laws at the Particle Data Group website.
The CPT Theorem

It can be shown that local relativistically invariant quantum field theories are also invariant under the combined CPT transformation. This result was proved independently by Pauli, Schwinger, Lüders, and Bell, see Wikipedia CPT theorem. A definitive proof incorporating earlier work was given by G. Lüders, Proof of the TCP theorem, Ann. Phys. (N.Y.) 2, 1–15 (1957).

Experimental searches for CPT violation in elementary particle decays have never found evidence for any breakdown of this theorem. Among the most precise measurements are bounds on particle-antiparticle mass differences, for example

\[
\left| \frac{m_{e^+} - m_{e^-}}{m_{\text{average}}} \right| < 8 \times 10^9, \quad \left| \frac{m_p - m_{\bar{p}}}{m_p} \right| < 2 \times 10^9, \quad \left| \frac{m_{K^0} - m_{\bar{K}^0}}{m_{\text{average}}} \right| < 6 \times 10^{-19},
\]

see http://pdg.lbl.gov/2013/tables/rpp2013-conservation-laws.pdf. The most sensitive are probably the searches for an electric dipole moment of the neutron, measured to be \( < 2.9 \times 10^{26} e\, \text{cm} \), and the electron \( < (10.5 \pm 0.07) \times 10^{28} e\, \text{cm} \), a nonzero value requiring both \( P \) and \( T \) violation.

A proof of the theorem can be constructed similarly to Pauli’s proof of the spin-statistics connection theorem. The steps are simpler because the transformations are discrete. This makes it unnecessary to make a Fourier transform to simplify the analysis of derivatives of the field operators.

The transformation properties of irreducible representations of the Lorentz group are defined separately under \( C, P, \) and \( T \). The most general forms of the Lagrangian density are then shown to be invariant under the combined CPT transformation. Transition amplitudes can then be shown to be invariant up to an overall phase factor. This procedure can easily be carried out explicitly for the quantized Klein-Gordon, Electromagnetic, and Dirac fields.
C, P, and T Symmetries of Dirac Fields

The transformation properties of the various fermion bilinears under $C$, $P$, and $T$ are summarized in the table below. Here we use the shorthand $(-1)^\mu \equiv 1$ for $\mu = 0$ and $(-1)^\mu \equiv -1$ for $\mu = 1, 2, 3$.

<table>
<thead>
<tr>
<th></th>
<th>$\bar{\psi}\psi$</th>
<th>$i\bar{\psi}\gamma^5\psi$</th>
<th>$\bar{\psi}\gamma^\mu\psi$</th>
<th>$\bar{\psi}\gamma^\mu\gamma^5\psi$</th>
<th>$\bar{\psi}\sigma^{\mu\nu}\psi$</th>
<th>$\partial_\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>+1</td>
<td>−1</td>
<td>$(-1)^\mu$</td>
<td>$-(-1)^\mu$</td>
<td>$(-1)^\mu(-1)^\nu$</td>
<td>$(-1)^\mu$</td>
</tr>
<tr>
<td>$T$</td>
<td>+1</td>
<td>−1</td>
<td>$(-1)^\mu$</td>
<td>$(-1)^\mu$</td>
<td>$-(-1)^\mu(-1)^\nu$</td>
<td>$-(-1)^\mu$</td>
</tr>
<tr>
<td>$C$</td>
<td>+1</td>
<td>+1</td>
<td>−1</td>
<td>+1</td>
<td>−1</td>
<td>+1</td>
</tr>
<tr>
<td>$CPT$</td>
<td>+1</td>
<td>+1</td>
<td>−1</td>
<td>−1</td>
<td>+1</td>
<td>−1</td>
</tr>
</tbody>
</table>

We have included the transformation properties of the tensor bilinear (see Problem 3.7), and also of the derivative operator.

[Summary from Peskin-Schroeder page 71]

If the $4 \times 4$ matrices representing space inversion and time reversal of $x^\mu$ give the identity when applied twice

$$PP = P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^2 = I, \quad T^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^2 = I.$$
The operations $C$, $P$, and $T$ are **idempotent**. Their representations on Fock space states and operators must have the same property. To implement space inversion for example an operator representation $\mathcal{P}$ must be found such that

$$\langle A|O|B\rangle = \langle A|\mathcal{P}^2 O \mathcal{P}^2 |B\rangle, \quad O \rightarrow \mathcal{P} O \mathcal{P}, \quad |A\rangle \rightarrow O|A\rangle$$

gives the appropriate transformation of Fock space operators $O$ and Fock space states $A, B$.

### Active and Passive Views of Spacetime Transformations

Peskin-Schroeder use the **active view**: a fixed observer moves the system and its fields

$$\psi(x) \rightarrow \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1} x).$$

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Figure 3.1. When a rotation is performed on a vector field, it affects the **orientation** of the vector as well as the location of the region containing the configuration.

Figure 11.4 Addition of velocities.
The **passive view**, in which many observers study the fixed system using different reference frames related by the symmetry transformation (Jackson Fig. 11.4), is more natural in quantum field theory because (1) there is one universe with one spacetime and one vacuum, and (2) some transformations like spacetime inversion cannot be performed physically.

\[
x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu, \quad \psi(x) \rightarrow \psi'(\Lambda x) = \Lambda_{1/2} \psi(x).
\]

The active and passive views are, of course, mathematically equivalent. Use your favorite view!

**Invariance of the Lagrangian Density**

The Lagrangian density for the Dirac field and its interactions

\[
L(x) = \bar{\psi}(x) \left[ i \gamma^0 \partial_t + i \gamma \cdot \nabla - m \right] \psi(x) - e \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) + \ldots
\]

must be invariant under symmetry transformations, including \(C, P\) and \(T\). This invariance determines how the \(4 \times 4\) Dirac matrix expression sandwiched between \(\hat{\psi} \cdots \hat{\psi}\) must transform. Using spacetime translation invariance, this transformation property can be determined at the origin \(x^\mu = 0\), which is invariant under rotations, boosts, spacetime inversions, and charge conjugation. For a local quantity at the origin, the \((x)\) dependence can be dropped and there is no difference between the active and passive views.

**Space Inversion and Parity**

Space inversion should invert the 3-momentum \(\mathbf{p}\) of a single particle state, which implies

\[
\mathcal{P} \hat{a}_\mathbf{p} \mathcal{P} = \eta_a \hat{a}_{-\mathbf{p}}, \quad \mathcal{P} \hat{b}_\mathbf{p} \mathcal{P} = \eta_b \hat{b}_{-\mathbf{p}},
\]
and $\eta_a^2 = \eta_b^2 = \pm 1$ so observables represented by Dirac bilinears are invariant. Then the Dirac field operator transforms according to

$$\mathcal{P} \hat{\psi}(t, \mathbf{x}) \mathcal{P} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( \eta_a \hat{a}_-^s \mathbf{u}^s(\mathbf{p}) e^{-iEt + i\mathbf{p} \cdot \mathbf{x}} + \eta_b \hat{b}_+^s \mathbf{v}^s(\mathbf{p}) e^{iEt - i\mathbf{p} \cdot \mathbf{x}} \right)$$

$$= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( \eta_a \hat{a}_-^s \mathbf{u}^s(\mathbf{p}) e^{-iEt + i\mathbf{p} \cdot \mathbf{x}} + \eta_b \hat{b}_+^s \mathbf{v}^s(\mathbf{p}) e^{iEt - i\mathbf{p} \cdot \mathbf{x}} \right)$$

$$= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( \eta_a \hat{a}_-^s \mathbf{u}^s(-\mathbf{p}) e^{-iEt - i\mathbf{p} \cdot \mathbf{x}} + \eta_b \hat{b}_+^s \mathbf{v}^s(-\mathbf{p}) e^{iEt + i\mathbf{p} \cdot \mathbf{x}} \right)$$

$$= \Gamma^P \hat{\psi}(t, -\mathbf{x}) ,$$

in the Peskin-Schroeder active view, where $\Gamma^P$ is a constant $4 \times 4$ Dirac matrix which represents the action of space inversion on the Dirac equations for the spinor wavefunctions

$$(\gamma \cdot \mathbf{p} - m) \mathbf{u}^s(\mathbf{p}) = 0 , \quad (\gamma \cdot \mathbf{p} + m) \mathbf{v}^s(\mathbf{p}) = 0 .$$

The conjugate field transforms as

$$\bar{\psi} = \hat{\psi}^\dagger \gamma^0 \rightarrow \hat{\psi}^\dagger \Gamma^P \gamma^0 = \bar{\psi} \gamma^0 \Gamma^P \gamma^0 .$$

Invariance of the Lagrangian density then requires

$$\gamma^0 \Gamma^P \gamma^0 \left[ i\gamma^0 \partial_t + i\mathbf{\gamma} \cdot \nabla - m \right] \Gamma^P = \left[ i\gamma^0 \partial_t + i\mathbf{\gamma} \cdot \nabla - m \right] ,$$
which determines the form of

\[ \Gamma^P = \eta_\psi \gamma^0, \quad \gamma^0 \gamma^0 \gamma^0 = \gamma^0, \quad \gamma^0 \gamma \gamma^0 = -\gamma, \quad \gamma^0 1 \gamma^0 = 1, \quad \eta_\psi \eta_\psi^* = 1, \]

which changes the sign of the 3-momentum gradient term but not the energy or mass terms. Using explicit expressions for the spinor wavefunctions in the Weyl representation, it is easy to verify that

\[ \gamma^0 u^s = u^s, \quad \gamma^0 v^s = -v^s, \]

which fixes the parity phases

\[ \eta_\psi = \eta_a = -\eta_b^*, \quad \eta_a \eta_b = -\eta_a^2 = -1, \quad \eta_\psi = \pm i. \]

The phase factor \( \eta_\psi \) contributes \( \eta_\psi \eta_\psi^* = 1 \) to bilinear observables of the form \( \bar{\psi} \Gamma \psi \).