1 Interaction of Quantum Fields with Classical Sources

A source is a given external function on spacetime $t, \mathbf{x}$ that can couple to a dynamical variable like a quantum field. Sources are fundamental in the functional and path integral formulations of field theory, which is developed in Peskin-Schroeder Chapter 9. For example, the scalar field propagator is given in Eqs. 9.34,35) as a functional derivative of a generating functional $Z[J]$ with respect to a classical source function $J(x)$

$$Z[J] \equiv \int \mathcal{D}\phi \exp \left[ i \int d^4x \left( \mathcal{L} + J(x)\phi(x) \right) \right],$$

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{1}{Z_0} \left( -i \frac{\delta}{\delta J(x_1)} \right) \left( -i \frac{\delta}{\delta J(x_2)} \right) Z[J] \Big|_{J=0}.$$

Interactions of quantum fields with classical sources are also very important in many applications. Examples include electromagnetic fields in laboratory experiments and the gravitational field in cosmology. Black hole thermodynamics involves quantizing the Standard Model fields in a strongly curved background metric.

Peskin-Schroeder discuss a simple application in the last section of Chapter 2 on Particle Creation by a Classical Source. The Lagrange density for the Klein-Gordon field is taken to be

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi)^2 - \frac{1}{2}m^2\phi^2 + J(x)\phi(x).$$

If the function $J(x)$ is turned on for a finite interval of time, it will excite the vacuum state of the field
and create particles. The total number of particles produced is given by

\[
\int dN = \int \frac{d^3p}{(2\pi)^3 2E_p} \left| \tilde{J}(p) \right|^2.
\]

They discuss an important QED example in §6.1 Soft Bremsstrahlung on page 177:

Suppose that a classical electron receives a sudden kick at time \( t = 0 \) and position \( x = 0 \), causing its 4-momentum to change from \( p \) to \( p' \). (An infinitely sudden change of momentum is of course an unrealistic idealization. The precise form of the trajectory during the acceleration does not affect the low-frequency radiation, however. Our calculation will be valid for radiation with a frequency less than the reciprocal of the scattering time.)

\[ \begin{array}{c}
p' \quad \text{sudden kick at time } t = 0, \\
\text{when particle is at } x = 0
\end{array} \]

We can find the radiation field by writing down the current of this electron, and considering that current as a source for Maxwell’s equations.

What is the current density of such a particle? For a charged particle at rest at \( x = 0 \), the current would be

\[
j^\mu(x) = (1, 0) \cdot e \delta^{(3)}(x)
\]

\[
= \int dt \, (1, 0) \cdot e \delta^{(4)}(x^\mu - y^\mu(t)), \quad \text{with } y^\mu(t) = (t, 0).
\]

From this we can guess the current for an arbitrary trajectory \( y^\mu(\tau) \):

\[
j^\mu(x) = e \int d\tau \frac{dy^\mu(\tau)}{d\tau} \delta^{(4)}(x^\mu - y^\mu(\tau)). \quad (6.3)
\]

Jackson discuss many applications in radiation and accelerator physics in Chapters 15 and 16.
2 Feynman Diagrams in Statistical Mechanics

The anharmonic oscillator Lagrangian is

\[ L(\phi, \dot{\phi}) = \frac{1}{2} \dot{\phi}^2 - \frac{m^2}{2} \phi^2 - \lambda \phi^4. \]

The quantum mechanical amplitude that the oscillator with field eigenvalue \( \phi(t_a) \) at time \( t_a \) will be found in field eigenstate \( \phi(t_b) \) at time \( t_b \) is given by

\[ Z = \sum_{\text{paths}} \exp \left[ \frac{i}{\hbar} S_{\text{path}} \right], \quad S_{\text{path}} = \int_{t_a}^{t_b} dt \, L(\phi, \dot{\phi}). \]

in Feynman’s path integral formulation of quantum mechanics.

The classical statistical mechanics of the anharmonic oscillator is obtained by continuing the path integral to imaginary time \( \tau = it \)

\[ S_E = \frac{1}{2} \int d\tau \left[ \left( \frac{d\phi}{d\tau} \right)^2 + m^2 \phi^2 + \lambda \phi^4 \right] \simeq \frac{\epsilon}{2} \sum_j \left[ \left( \frac{\phi_{j+1} - \phi_j}{\epsilon} \right)^2 + m^2 \phi_j^2 + \lambda \phi_j^4 \right]. \]

Note that this is the classical Hamiltonian for a 1-dimensional chain of coupled anharmonic oscillators associated with each of the time steps, and the path integral

\[ Z = \left( \prod_j \int_{-\infty}^{\infty} d\phi_j \right) \exp \left[ -\frac{S_E}{\hbar} \right] = \left( \prod_j \int_{-\infty}^{\infty} d\phi_j \right) \exp \left[ -\beta H \right] \xrightarrow{\epsilon \to 0} \int \mathcal{D}[\phi] \exp \{ -S_E[\phi]/(k_B T) \}. \]
3 The Ginzburg-Landau Theory of Superconductivity

The interaction of superconducting electrons with applied magnetic fields is another very important application of QED coupled to a classical source field generated by permanent magnets or macroscopic currents.

Superconductivity in metals and alloys can be modeled at the atomic level using Bardeen-Cooper-Schrieffer (BCS) Theory, and at the macroscopic level using the Ginzburg-Landau field theory. Similarly to the linear chain of oscillators and phonon field theory, BCS theory describes Cooper pairs of electrons that undergo Bose-Einstein condensation and Ginzburg-Landau theory describes the long wavelength excitations of the superconducting condensate.

Altland-Simons discuss the Ginzburg-Model in §6.4 Superconductivity. The free energy of the superconducting condensate in an external magnetic field is a very important and instructive example of a charged scalar field coupled to a classical electromagnetic source. Peskin-Schroeder discuss the general concepts on page 269 Landau Theory of Phase Transitions in Chapter 8.

Landau was awarded the 1962 Nobel Prize in Physics “for his pioneering theories for condensed matter, especially liquid helium”, and Ginzburg shared the 2003 Nobel Prize in Physics “for pioneering contributions to the theory of superconductors and superfluids” with Abrikosov and Leggett.

Abrikosov predicted the existence of Vortex lattices in Type II superconductors in a magnetic field. Leggett was awarded the prize for his work on the superfluid phase transition of liquid Helium 3. He gave the 2013 Rustgi Lecture at UB.
4 Ginzburg-Landau Model of Type II Superconductors in a Magnetic Field

The Ginzburg-Landau model is a statistical field theory with a complex scalar field \( \Psi(\mathbf{r}) \) which represents the number density \( n(\mathbf{r}) \) and current density \( J(\mathbf{r}) \) of the superconducting condensate of Cooper pairs

\[
\Psi(\mathbf{r}) = |\Psi(\mathbf{r})| e^{i\theta(\mathbf{r})}, \quad n(\mathbf{r}) = |\Psi(\mathbf{r})|, \quad J(\mathbf{r}) = \frac{i e^* \hbar}{2m^*} (\Psi^* \mathbf{D} \Psi - \Psi \mathbf{D} \Psi^*), \quad \mathbf{D} \equiv \nabla + ie^* \mathbf{A}
\]

where \( e^* = 2e \) is the magnitude of the electric charge and \( m^* \) is the effective mass of a Cooper pair of electrons, and \( \mathbf{A}(\mathbf{r}) \) is the electromagnetic vector potential of the applied magnetic field \( \mathbf{H} = \nabla \times \mathbf{A} \).

![Diagram of the magnetic phase diagram of a type II superconductor.](image)

FIG. 1. (Color online) Schematic magnetic phase diagram of a type II superconductor.
Free Energy and Second Order Phase Transition

The free energy of the superconductor at temperature $T$ in the external field $H$ determined by the Ginzburg-Landau energy density

$$
\mathcal{F} = \mathcal{F}_{\text{normal}} + \frac{1}{2m^*} |\mathbf{D}\Psi|^2 + \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{2} H^2,
$$

where $\mathcal{F}_{\text{normal}}$ is the free energy of the normal (non-superconducting) electrons. The thermodynamic properties of the system are determined by the partition function

$$
Z = \int D\Psi D\Psi^* \exp \left[ -\frac{1}{k_B T} \int d^3 r \mathcal{F} \right],
$$

which can be evaluated as a perturbation series using Feynman diagrams and Feynman rules. It has also been studied numerically, for example using Monte Carlo methods.

The Abrikosov vortex lattice is predicted to melt into a vortex liquid at the upper critical field $H_{c2}$, but it is not known whether the transition is first order with a finite latent heat, or second order with continuous free energy. A strictly two-dimensional lattice is predicted to undergo a [Kosterlitz-Thouless transition]. Another important observable is Abrikosov ratio, which is the ratio of the thermal average of anharmonic energy density to the harmonic energy density

$$
\beta_A(T) = \frac{\langle |\Psi|^4 \rangle}{\left[ \langle |\Psi|^2 \rangle \right]^2}.
$$
The following show Monte Carlo results from Y. Kato and Phys. Rev. B 48, 7383–7391 (1993) compared with Feynman diagram results:

The problem of whether the transition is first order, second order, or Kosterlitz-Thouless has not been definitively settled.

FIG. 1. The Abrikosov ratio $\beta_A$ as a function of the reduced temperature $t$. The triangles, squares and circles are our Monte Carlo data for $N_s = 4^2$, $8^2$, and $12^2$, respectively. The sample-size dependence is negligible and all the data are in excellent agreement with the high-temperature expansion combined with [5,5] Borel-Padé analysis (solid curve) (Ref. 11).

FIG. 4. The normalized internal energy $\langle H \rangle / \langle H_{MF} \rangle$ as a function of the reduced temperature $t$ with the enlarged scale. The hysteresis behavior is observed between the heating (open circles) and cooling (solid circles) processes.
Feynman Diagrams for the Free Energy

*Critical behaviour of type II superconductors near* $H_{c_2}$

\[\text{Figure 1. Lowest order Feynman graphs. (a) first order, (b) and (c) second order.}\]

\[\text{Figure 2. Feynman Graphs without loops. (a) and (b) third order, (c), (d), (e), (f) and (g) fourth order.}\]


As we have computed it, the $f_{2D}$ series out to fourteen terms with rational coefficients is:

$$f_{2D}(x) = -2x^1 - 1x^2 + \left(\frac{38}{9}\right)x^3 - \left(\frac{1199}{30}\right)x^4 + \left(\frac{8166946}{17325}\right)x^5 - \left(\frac{94158226189}{14549535}\right)x^6 +$$

$$+ \left(\frac{50899900161327794214}{50256949018000725}\right)x^7 - \left(\frac{4151736434026261069431857966288755223}{2332699888866080474081039435100}\right)x^8$$

$$+ \left(\frac{344471807450518455084883998069313941551306507458648436919783473492126}{99245617213562927471771779898320841272897130218292215077771375}\right)x^9$$

$$+ \cdots + \left(\frac{n_{14}}{d_{14}}\right)x^{14} + \left(\frac{n_{15}}{d_{15}}\right)x^{15} + \cdots$$

Here $f_{2D}(x)$ is the free energy of the two-dimensional vortex lattice in terms of a dimensionless variable

$$x = \frac{\beta e^* HT}{2\pi L_z \tilde{\alpha}^2}, \quad \tilde{\alpha}(1 - 4x)^3 = \alpha + \frac{e^* H}{2m^*}.$$
References
