1 Vacuum Energy and Cosmology

The Fock-space representation of the Hamiltonian operator

$$\hat{H} = \int d^3x \frac{E^2 + B^2}{2} = \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3} \omega \left[ \hat{a}^\dagger(k, \lambda) \hat{a}(k, \lambda) + \frac{1}{2} \int d^3x \right], \quad \omega = |k|$$

has vacuum expectation value

$$\langle 0 | \hat{H} | 0 \rangle = V \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3} \frac{\omega}{2} = \frac{V}{8\pi} \lim_{|k|_{\text{max}} \to \infty} |k|_{\text{max}}^4, \quad V = \int d^3x$$

This is a rapidly increasing function of the maximum photon energy $\omega_{\text{max}} = |k|_{\text{max}}$ in the mode expansion. The ground state energy is quartically ultraviolet divergent. Most laboratory experiments only measure energy differences, and the predictions of QED agree extremely well with observations if all energies are measured relative to the same vacuum. One simply subtracts the infinite vacuum energy from all theoretical energy predictions.

Vacuum energy, like any other form of energy, will couple to the gravitational field. The effects vacuum fluctuation are important in cosmology. They potentially contribute to the cosmological constant in General Relativity and dark energy, which might be driving the accelerating expansion of the universe observed in supernova redshift data. These problems are reviewed by Weinberg[1] and Peebles & Ratra[2].
2 The Casimir Effect

The Casimir Effect predicts a force between two conductors separated by vacuum and caused by quantum fluctuations of the electromagnetic field. It is discussed in Altland-Simons §1.5 and Sakurai-Napolitano §7.6. It has been measured experimentally, for example by Mohideen and Roy[3] using the setup in the figure:

![Schematic diagram of the experimental setup.](image1)

FIG. 1. Schematic diagram of the experimental setup. Application of voltage to the piezo results in the movement of the plate towards the sphere. The experiments were done at a pressure of 50 mTorr and at room temperature.

![Scanning electron microscope image of the metallized sphere mounted on an AFM cantilever.](image2)

FIG. 2. Scanning electron microscope image of the metallized sphere mounted on an AFM cantilever.

The metal sphere of radius $R = 98 \mu m$ is separated from a metal plate by a distance $d \ll R$ and an attractive force between them is measured experimentally using a cantilever balance. The Casimir force
given theoretically by

\[
F_{cp}(d) = -\frac{\pi^2}{360} R \frac{\hbar c}{d^3} \left[ 1 - 4 \frac{c}{\omega_p d} + \frac{72}{5} \left( \frac{c}{\omega_p d} \right)^2 \right] \left[ 1 + 6 \left( \frac{A_r}{d} \right)^2 \right] \left[ 1 + \frac{720}{\pi^2} f(\xi) \right],
\]

where the terms in brackets are small corrections due to conduction electron fluctuations at the metal surfaces, the roughness of the surfaces, and a finite temperature correction.

**Casimir Force between Parallel Plates in Vacuum**

Experiments are done with a sphere because it is hard to maintain a constant separation between two flat surfaces. It is much easier theoretically to compute the Casimir force between two parallel plates. The
calculation was done by Casimir in 1948 for parallel plates of area $A$ separated by distance $d$ with the prediction

$$F(d) = -\frac{\pi^2 \hbar c}{240} \left( \frac{A}{d^4} \right) = -\left(1.30 \times 10^{-19} \text{ N/m}^2\right) \times \left( \frac{A}{d^4} \right).$$

This effect predicted by Casimir was first observed by Sparnaay\[4\], who measured an attractive force between metal plates with magnitude that varied between $1 - 4 \times 10^{-19} \text{ N/m}^2$ as the plate separation $d$ was varied between 2 and 10 $\mu$m.

**Casimir’s Assumptions and Mode Expansion**

The energy of the vacuum due to fluctuations in the electromagnetic field is

$$\langle 0 | \hat{H} | 0 \rangle = V \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3} \frac{\omega}{2}.$$

Casimir considered modifying the vacuum state by introducing two macroscopic but finite parallel conducting plates of area $A = L^2$ separated by a distance $d \ll L$. The infinite energy of the vacuum is due to the ground state energies of the the photon field operators at each point in space. The vaccum is a single entangled quantum state of all of these oscillators, and every one of them will be affected by the introduction of the plates. However, it is reasonable to assume that this effect falls off with distance from the plates and that the vacuum energy density far from the plates will approximate that of the true vaccuum.

Casimir calculated the change in vaccuum energy due to the plates based on two assumptions: (1) The pho-
ton modes between the plates and propagating normal to them obey the classical electrodynamic boundary conditions at the conducting surfaces, and (2) the modes propagating inside and parallel to the plates and all modes outside the plates are not changed. This is similar to assuming the electric field inside a parallel plate capacitor is normal to the plates and neglecting edge effects at the edges of the plates and the electric field outside, which gives a good first approximation for the capacitance.

First, consider the energy of the vacuum in the presence of the plates. The boundary conditions at the surface of the plates are satisfied if the normal component of the vector potential vanishes at the surface. Modes which satisfy this condition are standing waves with normal wavevector component $k_\perp = n\pi/d, n = 1, 2, \cdots$, and the sum of zero-point energies with the normal components discrete and the components parallel to the plates continuous is

$$\sum \frac{\omega^2}{2} \to \sum_{\lambda=1}^{2} A \int \frac{d^2k_\parallel}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{2} \sqrt{k_\parallel^2 + \frac{n^2\pi^2}{d^2}}.$$

The modes with $k_\perp = 0$ must also be taken into account. These are plane wave propagating parallel to the plates. To satisfy the boundary conditions, only one polarization state with electric field normal and magnetic field parallel to the surface is allowed, contributing

$$\sum \frac{\omega^2}{2} \to A \int \frac{d^2k_\parallel}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{|k_\parallel|}{2}.$$

to the energy between the plates. The vacuum energy in the region of volume $V = Ad$ between the plates
can also be decomposed into normal and parallel contributions

\[ \langle 0 | \hat{H} | 0 \rangle = V \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^3} \frac{\omega}{2} \]

\[ = \sum_{\lambda=1}^{2} A \int \frac{d^2k_\parallel}{(2\pi)^2} \times d \int_{-\infty}^{\infty} \frac{dk_{\perp}}{2\pi} \sqrt{\frac{k^2_{\parallel} + k^2_{\perp}}{2}} \]

\[ = \sum_{\lambda=1}^{2} A \int \frac{d^2k_\parallel}{(2\pi)^2} \int_{0}^{\infty} dn \sqrt{\frac{k^2_{\parallel} + n^2\pi^2}{d^2}} \text{, where } n = \frac{d}{\pi} \times |k_{\perp}| \].

The energy difference per unit area between the plates is

\[ \mathcal{E} = \frac{\sum \omega_{\parallel} \text{plates} - \langle 0 | \hat{H} | 0 \rangle}{A} = \frac{1}{A} \int_{0}^{\infty} dk \ k \left( \frac{k}{2} + \sum_{n=1}^{\infty} \sqrt{\frac{k^2 + n^2\pi^2}{d^2}} - \int_{0}^{\infty} dn \sqrt{\frac{k^2 + n^2\pi^2}{d^2}} \right) \].

### Regulating the Ultraviolet Divergences

The sum and integral over \( n \) in the expression for \( \mathcal{E} \) are separately divergent. If \( \mathcal{E} \) is to be finite, the sum of divergent terms in parentheses must be finite and tend to zero faster than \( 1/k^2 \) for the integral over \( k \) to converge. Sakurai-Napolitano (page 476) reference “Quantum Field Theory” by Itzykson and Zuber for way to take the limit based on physical arguments: Assume that the classical field boundary condition approximation must be modified for wavenumbers larger than \( \pi/a_0 \) where \( a_0 \) is the Bohr radius. The metal
plates effectively become transparent to modes with much larger wavenumbers and the sum and integral over \( n \) cancel exactly. This effect can be achieved cutting off the wavenumbers \( k \) and \( \pi n/d \) with a smooth function \( f(k) \).

The ultraviolet divergences arise from representing spatial points by real numbers in \( \mathbb{R}^3 \), which has a continuously infinite number of points in any finite volume. In computational science applications real variables are represented by a finite set of real fixed precision numbers. In lattice gauge theory, spacetime is represented as a finite regular hypercubic lattice of points. An example of a function \( f(k) \) based on a finite regular cubic lattice approximation is

\[
\begin{align*}
\mathbf{k}^2 e^{i\mathbf{k} \cdot \mathbf{x}} &= -\nabla^2 e^{i\mathbf{k} \cdot \mathbf{x}} \rightarrow \sum_{j=1}^{3} \frac{2 - e^{i k_j a} - e^{-i k_j a}}{a^2} e^{i\mathbf{k} \cdot \mathbf{x}} = \sum_{j=1}^{3} \frac{4}{a^2} \sin^2 \left( \frac{k_j a}{2} \right) e^{i\mathbf{k} \cdot \mathbf{x}} = f(k) e^{i\mathbf{k} \cdot \mathbf{x}} .
\end{align*}
\]

The lattice provides a cutoff on the wavenumber components \( |k_j| \leq \pi/a \). The lattice spacing \( a \) is chosen so that \( d/a \) is an integer and the plates coincide with lattice planes in the transverse direction. The lattice spacing

Itzykson and Zuber use a cutoff function \( f(|k|) \), which we can take for example to be

\[
f(|k|) = \frac{2}{a} \left| \sin \left( \frac{|k| a}{2} \right) \right| \theta(\pi - |k| a) \rightarrow_{a \to 0} \sqrt{\mathbf{k}^2 + k_\perp^2} ,
\]

with the cutoff \( a \) chosen so that \( d \gg a_{0,\text{Bohr}} \gg a \). Changing integration variable to

\[
u = \frac{\mathbf{k}^2 d^2}{\pi} , \quad \int_{0}^{\infty} d\mathbf{k} k = \frac{\pi}{2d^2} \int_{0}^{\infty} du \]

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the energy difference per unit area becomes
\[ E = \frac{\pi^2}{4d^3} \int_0^\infty du \left[ \frac{\sqrt{u}}{2} f \left( \frac{\pi \sqrt{u}}{d} \right) + \sum_{n=1}^{\infty} \sqrt{u + n^2} f \left( \frac{\pi \sqrt{u + n^2}}{d} \right) - \int_0^\infty dn \sqrt{u + n^2} f \left( \frac{\pi \sqrt{u + n^2}}{d} \right) \right] \]
\[ = \frac{\pi^2}{4d^3} \left[ \frac{1}{2} F(0) + F(1) + F(2) + \cdots - \int_0^\infty dn \ F(n) \right], \]
with the definition
\[ F(n) \equiv \int_0^\infty du \sqrt{u + n^2} f \left( \frac{\pi \sqrt{u + n^2}}{d} \right). \]
Interchanging the \( u \) integration with the \( n \) sum and integral is allowed because the cutoff function makes them absolutely convergent. As \( n \to \infty \), the function \( F(n) \to 0 \), and the Euler-Maclaurin formula gives
\[ \frac{1}{2} F(0) + F(1) + F(2) + \cdots - \int_0^\infty dn \ F(n) = \frac{1}{2!} B_2 F'(0) - \frac{1}{4!} F'''(0) + \cdots \]
in terms of the Bernoulli numbers defined by the series
\[ \frac{y}{e^y - 1} = \sum_{\ell=0}^{\infty} B_{\ell} \frac{y^\ell}{\ell!}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \ldots \]
and
\[ F(n) = \int_{n^2}^\infty du \sqrt{u} F \left( \frac{\pi \sqrt{u}}{d} \right), \quad F'(n) = -2n^2 f \left( \frac{\pi n}{d} \right). \]
The function \( f \) can be chosen so \( f(0) = 1 \) that all of its derivatives vanish at the origin. Then
\[ F'(0) = 0, \quad F'''(0) = -4. \]
and higher derivatives of $F$ vanish. The cutoff $a$ has disappeared from the final result energy relative to vacuum and force per unit area between the plates is

$$
\mathcal{E} = \frac{\pi^2 B_4}{4! d^3} = -\frac{\pi^2}{720 d^3}, \quad \mathcal{F} = -\frac{\text{d}\mathcal{E}}{\text{d}d} = -\frac{\pi^2}{240 d^4}.
$$
References


