1 Equal-time and Time-ordered Green Functions

Predictions for observables in quantum field theories are made by computing expectation values of products of field operators, which are called Green functions or correlation functions.

The two most important types of correlation functions are the vacuum (ground state) expectation values of (1) products of operators at one fixed instant of time called equal-time Green functions, and (2) products of operators at different instants of time ordered by their time values and called time-ordered Green functions. Equal-time Green functions are used to compute the properties of ground states and bound states in quantum field theory, many body theory, and equilibrium statistical mechanics. Time-ordered Green functions are most useful in scattering problems: the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula (Peskin-Schroeder §7.2) converts time-ordered Green functions to S-matrix elements which determine scattering cross sections.

Feynman diagrams and Feynman rules are very useful in evaluating any Green function in perturbation theory. Most of the important concepts and derivations can be developed in the one-dimensional anharmonic oscillator chain. The continuum limit of the chain describes phonons in condensed matter, and Klein-Gordon scalar particles in quantum field theory. The quantum Hamiltonian density and equal-time canonical commutator for the field and its canonical momentum are

\[
\hat{H}(\hat{\phi}, \hat{\pi}) = \frac{1}{2M} \hat{\pi}^2 + \frac{k_s a^2}{2} (\partial_x \hat{\phi})^2 + \frac{m^2}{2} \hat{\phi}^2 + \lambda \hat{\phi}^4 , \quad \left[ \hat{\phi}(t, x), \hat{\pi}(t, x') \right] = i \delta(x - x') .
\]

The generalizations to 4-dimensional spacetime are straightforward.
The Feynman Propagator

The most basic time-ordered Green function for the Klein-Gordon field is the Feynman propagator

$$\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle = D_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i}{p^2 - m^2 + i\epsilon}.$$ 

Time-ordering $T(\ldots)$ is defined by

$$T(\phi(x)\phi(y)) = \phi(x)\phi(y)\theta(x^0 - y^0) + \phi(y)\phi(x)\theta(y^0 - x^0), \quad \theta(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

The Feynman $i\epsilon$ in the denominator $\epsilon \to 0^+$ fixes the boundary conditions for the Green function, which can be evaluated by contour integration using the residue theorem at the two poles $p^0 = \pm E_p = \pm \sqrt{p^2 - m^2}$

The propagator represents the probability amplitude for a virtual Klein-Gordon particle to propagate in spacetime between points $x$ and $y$. The Feynman boundary conditions correspond to positive energy modes with $p^0 > 0$ propagating forward in time and negative energy modes with $p^0 < 0$ propagating backward in time.
2 Fock Space and Occupation Number Representation

A problem with the operators \( \hat{\phi}(t, x) \), \( \hat{\pi}(t, x') \) at a given time \( t \) is that they are parametrized by the real numbers \( x, x' \in \mathbb{R} \), which cannot be counted. Observations on physical systems measure a countable finite number of real values. To calculate observables in a field theory and compare with experimental measurements, we need to choose a sufficiently large number of parameter values so that any measurement can be predicted to within experimental uncertainty.

The most widely used method for choosing a countable subset of \( \mathbb{R} \) is a Fourier representation on a finite interval \( L \)

\[
\hat{\phi}(t, x) = \frac{1}{\sqrt{L}} \sum_{k} e^{ikx} \hat{\phi}_k(t), \quad k = \frac{2\pi n}{L}, n = 0, \pm 1, \pm 2, \ldots, \quad \hat{\phi}_k(t) = \frac{1}{\sqrt{L}} \int_{0}^{L} dx e^{-ikx} \hat{\phi}(t, x).
\]

In a classical field theory, this restricts the solution space to periodic piece-wise continuous and square-integrable functions. As \( L \to \infty \) calculated observables can develop singularities called infrared divergences. The infinite number of Fourier modes as \( k \to \pm \infty \) can cause singularities called ultraviolet divergences. For a theory to be useful, any observable that can be measured experimentally must be computable and well-defined in the infrared and ultraviolet limits.

For many applications, it is possible to work with continuous Fourier modes. These include most applications involving perturbation theory and Feynman diagrams. In strongly-coupled field theories, perturbation theory can break down. The most widely used method to tackle such problems is to discretize \( x \) with a finite set of lattice of points \( x \in \{ ja, j = 1, \ldots, L/a \} \) and an equal number of Fourier modes.
Plugging the Fourier series for $\hat{\phi}$ and $\hat{\pi}$ in the Hamiltonian density, integrating over the interval $L$ and using the orthogonality of Fourier modes

$$\frac{1}{L} \int_0^L dx \ e^{ikx} = \delta_{k,0} ,$$

the quantum Hamiltonian for the finite mode set becomes

$$\hat{H} = \int_0^L dx \ \hat{\mathcal{H}} = \sum_k \left[ \frac{1}{2M} \hat{\pi}_k \hat{\pi}_{-k} + \frac{M \omega_k^2}{2} \hat{\phi}_k \hat{\phi}_{-k} + \frac{m^2}{2} \hat{\phi}_k \hat{\phi}_{-k} \right] + \frac{\lambda}{L} \sum_{k_1,k_2,k_3} \hat{\phi}_{k_1} \hat{\phi}_{k_2} \hat{\phi}_{k_3} \hat{\phi}_{-k_1-k_2-k_3} ,$$

where $\omega_k = v |k|$ and $v = a \sqrt{k_s/M}$ is the classical sound velocity. The quadratic Hamiltonian is a sum over mode pairs with wavenumber $\pm k$. It can be completely diagonalized by introducing ladder operators

$$\hat{a}_k = \sqrt{\frac{ME_k}{2}} \left( \hat{\phi}_k + \frac{i}{ME_k} \hat{\pi}_k \right) , \quad \hat{a}^\dagger_k = \sqrt{\frac{ME_k}{2}} \left( \hat{\phi}_{-k} - \frac{i}{ME_k} \hat{\pi}_k \right) , \quad E_k^2 = \omega_k^2 + m^2$$

with commutation relations

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'} , \quad [\hat{a}_k, \hat{a}'_{k}] = 0 , \quad [\hat{a}^\dagger_k, \hat{a}'_k] = 0 .$$

The quadratic Hamiltonian is

$$\hat{H}_0 = \sum_k \left[ \frac{1}{2M} \hat{\pi}_k \hat{\pi}_{-k} + \frac{M \omega_k^2}{2} \hat{\phi}_k \hat{\phi}_{-k} + \frac{m^2}{2} \hat{\phi}_k \hat{\phi}_{-k} \right] = \sum_k E_k \left( a_k^\dagger a_k + \frac{1}{2} \right) ,$$

which is a sum of independent simple harmonic oscillator Hamiltonians.
Setting $m^2 = 0, \lambda = 0, E_k = \omega_k$ gives the quantum chain discussed in Altland-Simons §1.4. Setting $M = 1, v = c = 1$ gives a discrete version of the one-dimensional Klein-Gordon quantum field theory in the continuum limit $L \to \infty$ and $a \to 0$. The finite mode set Klein-Gordon theory can be studied numerically on a computer using the methods of Lattice Gauge Theory, which is explained in Kogut’s review article[2].

The Hilbert space of $\hat{H}_0$ can be described rigorously. It is the direct product of the Hilbert spaces of a finite number of non-interacting quantum oscillators with frequencies $\omega_k$. Each such space is spanned by
the eigenstates $|n_k\rangle$ of the corresponding quantum oscillator. The states of the chain

$$|n_1, n_2, \ldots\rangle = \prod_k \frac{1}{\sqrt{n_k!}} (a^\dagger_k)^{n_k} |0\rangle$$

are labeled by sets $\{n_1, n_2, \ldots\}$ of non-negative integers. The integer $n_k$ can be interpreted as the number of quasi-particles with energy $\omega_k$. The state $|0\rangle = |0, 0, \ldots\rangle$ with all $n_k = 0$ is called the quasi-particle vacuum state. By construction, the quasi-particles are identical and obey Bose-Einstein statistics.

Altland-Simons §2.1 has a more detailed discussion of the properties of this Fock space. It has been constructed here in a special way using the properties of non-interacting quantum harmonic oscillators. It can be proved rigorously that this constructed Fock space is a complete irreducible representation up to a unitary transformation of the Hilbert space of the original Hamiltonian. This result is crucial for a consistent quasi-particle interpretation. The Hamiltonian $\hat{H}_0$ represents non-interaction quasi-particles. The occupation quantum numbers $n_k$ cannot change. Interactions, represented for example the $\lambda \phi^4$ term in the Hamiltonian density, can simultaneously create and annihilate quasi-particles with different $\omega_k$. The Stone-von Neumann Theorem (see Altland-Simons §2.4 Problems) ensures that quasi-particles with different $\omega_k$ are all identical bosons.
3 Quantum Oscillator on a Spacetime Lattice

A simple way of deriving the Feynman rules for the anharmonic oscillator

\[ L(\phi, \dot{\phi}) = \frac{1}{2} \dot{\phi}^2 - \frac{m^2}{2} \phi^2 - \lambda \phi^4. \]

is to start with Feynman’s path integral formula for Green functions.

§2.6 of Sakurai-Napolitano discusses propagators in quantum mechanics and introduces the Feynman path integral\[1\]. It is useful to review this method for a single quantum oscillator. There is a good discussion in §III-B pages 664-666 of Kogut’s review\[2\] which makes use of a lattice in time. On a discrete time lattice, time evolution is effected by a transfer matrix. Analytic continuation to imaginary time gives the partition function of of an ensemble of classical oscillators at finite temperature.

To switch to Kogut’s notation the classical Lagrangian for the harmonic oscillator is

\[ L_0(\phi, \dot{\phi}) \rightarrow \mathcal{L}(x, \dot{x}) = \frac{1}{2} (\dot{x}^2 - \omega^2 x^2). \]

The amplitude that the particle will be initially at \( \phi(t_a) = (x_a, t_a) \) and finally at \((x_b, t_b)\) is given by

\[ Z = \sum_{\text{paths}} \exp \left[ \frac{i}{\hbar} S_{\text{path}} \right], \quad S_{\text{path}} = \int_{t_a}^{t_b} dt \mathcal{L}. \]
This amplitude can be computed explicitly

\[ Z = \sqrt{\frac{\omega}{2\pi i\hbar \sin(\omega T)}} \exp \left[ \frac{i\omega}{2\hbar \sin(\omega T)} \left\{ (x_a^2 + x_b^2) \cos(\omega T) - 2x_a x_b \right\} \right], \quad T = t_b - t_a. \]

An exact closed form solution has never been found for the anharmonic oscillator

\[ \mathcal{L}(x, \dot{x}) = \frac{1}{2} \left( \dot{x}^2 - \omega^2 x^2 \right) - \lambda x^4. \]

The path integral provides an easy derivation of the Feynman rules for the asymptotic perturbation series. To define the sum over paths, discretize time as shown in the figure, and convert the oscillating exponential into a damped exponential by analytic continuation to imaginary time \( \tau = it \).

**FIG. 4.** A discrete time axis and illustrative path in quantum mechanics.
This converts 2-dimensional Minkowski “spacetime” \((t, x)\) to Euclidean space \((\tau, x)\) with Euclidean action

\[
S_E = \frac{1}{2} \int d\tau \left[ \left( \frac{dx}{d\tau} \right)^2 + \omega^2 x^2 + \lambda x^4 \right] \simeq \frac{\epsilon}{2} \sum_j \left[ \left( \frac{x_{j+1} - x_j}{\epsilon} \right)^2 + \omega^2 x_j^2 + \lambda x_j^4 \right].
\]

Note that this is the classical Hamiltonian for a 1-dimensional chain of coupled anharmonic oscillators associated with each of the time steps, and the path integral

\[
Z = \left( \prod_j \int_{-\infty}^{\infty} dx_j \right) \exp \left[ -\frac{S_E}{\hbar} \right] = \left( \prod_j \int_{-\infty}^{\infty} dx_j \right) \exp \left[ -\beta H \right]
\]

is the configuration space partition function of a canonical ensemble of classical anharmonic oscillators at temperature \(\hbar = 1/\beta\).
References
