1 Transformations and Symmetries

Fields are functions of spacetime points $x$. A spacetime point $x = (ct, \mathbf{r})$ is a set of 4 real numbers assigned to an event by an observer in a particular reference frame. In a relativistically invariant theory, observers in different inertial reference frames related by Poincaré transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu,$$

should measure equivalent results in agreement with equivalent equations that are also related by Poincaré transformations.

Passive View of a Symmetry Transformation

In non-relativistic quantum mechanics, symmetries transformations are usually defined using an active view: in a translation, the experimeter moves the atom from point $A$ to point $B$. In field theory, the systems is the manifold of events, including the vacuum which is infinite in extent. It is more natural to think of two experimenters, one with origin at point $A$ and the other at point $B$, observing the same manifold of events. This is the passive view.

In the passive view, the system is the same and must have the equivalent values for measured by all observers. A scalar field $\phi(x)$ must have the same functional form and the same numerical values at the same spacetime point with equivalent coordinates

$$\phi'(x') = \phi(x).$$
The electromagnetic field $A^\mu(x)$ has four components representing a 4-vector at each spacetime point. In the passive view, the magnitude and direction do not change, but the numerical values of its components in rotated or boosted reference frames will change because the coordinate axes point in different directions

$$A'^\mu(x') = \Lambda^\mu_\nu A^\nu(x).$$

The scalar and electromagnetic fields are finite-dimensional irreducible representations of the Poincaré group of transformations at each spacetime point. The field $\phi(x)$ has one component function and the representation is one-dimensional transforming through multiplication by the number 1. The field $A^\mu(x)$ is a 4-component vector representation with components arranged as a column vector transforming through multiplication by the $4 \times 4$ matrix $\Lambda^\mu_\nu$. The representation is irreducible because no subset of components is invariant under all transformations.

The irreducible representations of the Poincaré group by fields with a finite number of components $\psi_\alpha(x)$, $\alpha = 1, \ldots, n$ that can be used to construct quantum field theories were studied by Wigner in a rigorous mathematical paper based on work by Dirac, Majorana and others. He classified the representations as time-like corresponding to particles with mass, and light-like corresponding to zero mass particles. The first step is to expand the field in plane waves with fixed momentum

$$\psi_\alpha(x) = \int \frac{d^4p}{(2\pi)^4} \delta^4(p^2 - m^2) e^{-ip \cdot x} \phi_\alpha(p).$$

The Lorentz transformations which do not change the momentum $p^\mu$ form a subgroup called the little group of $p$, which is represented by $n \times n$ unitary matrices $D_{\alpha\beta}$

$$\phi'_\alpha(p) = D_{\alpha\beta} \phi_\beta(p), \quad D^*_{\gamma\alpha} D_{\gamma\beta} = \delta_{\alpha\beta}.$$
2 Representations of the Lorentz Group

Wigner's construction is most useful for canonical quantization of a field using the Hamiltonian formulation. In a Lagrangian formulation, it is more convenient to work with the most general field functions not constrained by the Euler-Lagrange equations to be “on-shell” with definite mass and momentum. The finite-dimensional irreducible representations of full Lorentz group can also be determined.

The spatial rotations and boosts of the spacetime vector $\mu$ form a group called the proper orthochronous Lorentz group isomorphic to the Lie group $SO^+(1,3)$. Proper means spatial reflections through the origin are excluded, and orthochronous means that time-reversal is also excluded. A Lorentz transformation is parametrized by spatial rotations about three independent axes, and boosts along three independent spatial directions. The rotations can be parametrized by a 3-component vector $\omega$ with $|\omega| \leq \pi$, and the boosts can be parametrized by a three component vector $\zeta$ with $|\zeta| < \infty$. A $4 \times 4$ matrix representation of Lorentz transformations of $x^\mu$ can be written

$$\Lambda(\omega, \zeta) = \exp \left[-J \cdot \omega - K \cdot \zeta\right],$$

where the six matrices $J, K$ are the generators of the Lie algebra $so(1, 3)$ of the group

$$[J^i, J^j] = -\epsilon^{ijk} J_k, \quad [J^i, K^j] = -\epsilon^{ijk} K_k, \quad [K^i, K^j] = +\epsilon^{ijk} J_k,$$

where the 3-dimensional Levi-Civita tensor is totally antisymmetric with components

$$\epsilon^{ijk} = -\epsilon_{ijk}, \quad \epsilon^{123} = +1, \quad \epsilon^{12k} J_k = \epsilon^{123} J_3 = -J^3.$$
Explicit forms for the matrices are given in Jackson Chapter 11

\[
J_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad J_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
K_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K_3 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

The real pseudo-orthogonal matrices \( \Lambda \) with \( \det \Lambda = 1 \) define the 6-parameter Lie group \( SO^+(1,3) \). Pseudo means matrix multiplication includes the metric \( g_{\mu \nu} \). The indefinite orthogonal group \( O(p,q) \) is the group of real transformations on \( \mathbb{R}^{p+q} \) which leaves the indefinite quadratic form \( x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \) invariant. \( SO^+(p,q) \) is the special (S) connected (+) subgroup with unit determinant. The group \( O(1,3) \) has 4 disconnected components, which can be generated by the 4-element discrete group including space reflection and time reversal

\[
I = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad T = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad PT = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Note that \( \det PT = +1 \) but its component is disconnected from the identity component \( SO^+(1,3) \) containing the unit element \( I \).
Finite Dimensional Representations of the Lorentz Algebra

The group manifold of \( \text{SO}^+(1, 3) \) has two components. The rotations can be parametrized by the points in a ball (solid sphere) in \( \mathbb{R}^3 \) with \( |\omega| \leq \pi \). This is a compact manifold with finite volume \( 4\pi^4/3 \text{ rad}^3 \). However, it is multiply connected because every point on the closed surface represents the same rotation as the diametrically opposite point. The group manifold of the boosts is the whole of \( \mathbb{R}^3 \), which is non-compact with infinite volume. The multiple-connectedness and non-compact nature of the combined manifold make it difficult to analyze mathematically.

The standard mathematical procedure to deal with these problems is to complexify the Lie algebra of generators. In this case, the complexification

\[
A = \frac{J + iK}{2} , \quad B = \frac{J - iK}{2} ,
\]

yields a Lie algebra with imaginary structure constants

\[
A \times A = iA , \quad B \times B = iB , \quad [A_j, B_k] = 0 ,
\]

which is essentially two independent copies of the spin one-half algebra of the Pauli matrices

\[
\left[ \frac{\sigma}{2}, \frac{\sigma}{2} \right] = i\frac{\sigma}{2} , \quad \sigma = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} .
\]

More precisely the complexified algebra \( \mathfrak{so}(1, 3)_C \) is isomorphic to direct sum \( \mathfrak{sl}(2, C') \oplus \mathfrak{sl}(2, C') \). The representations of \( \mathfrak{sl}(2, C') \) are essentially the same as those of \( \mathfrak{su}(2) \). Finally, the group SU(2) is the
covering group of SO(3) with a closed (simply connected with no boundary) group manifold twice the volume of SO(3). Peterson’s article[3] gives 3 ways of visualizing the double cover of SO(3) and an interesting interpretation of Dante’s *Paradiso*.

The goal of complexification is to relate a general Lie algebra to the simple Lie algebras that were completely classified by Cartan. The Lie algebras $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ correspond to the algebra $A_1$ of the $A$ series in Cartan’s classification, see Figure 1.

The algebra $\mathfrak{su}(2)$ is the simplest in Cartan’s method, which is a generalization of finding eigenvalues and eigenvectors in quantum mechanics. A Cartan subalgebra is a maximum set of commuting generators. The number of commuting generators gives the rank of the algebra and the number of circles in the Dynkin
The Lie algebra $\mathfrak{su}(2)$ has rank 1 and its Cartan subalgebra can be chosen to be $\sigma^3/2$ with eigenvalues $\pm 1/2$. The irreducible representations are the familiar angular momentum states of quantum mechanics with spin $j = 0, 1/2, 1, 3/2, 2, \ldots$ and dimension $2j + 1$.

The irreducible representations of the Lorentz algebra $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ are the direct products of two independent $\mathfrak{su}(2)$ algebras corresponding to the generators $A$ with spin $j_+ = 0, 1/2, \ldots$ and $B$ with spin $0, 1/2, \ldots$. The representation $(j_+, j_-)$ has dimension $(2j_+ + 1)(2j_- + 1)$.

A scalar field transforms according to the representation $(0, 0)$, a vector field according to $(1/2, 1/2)$ with $2 \times 2 = 4$ components, etc. The Lorentz transformation matrices of the vector representation are real and the field can be taken to be real, for example the real electromagnetic potential field $A_\mu(x)$. The $(1, 1)$ representation is real with dimension $3 \times 3 = 9$, and gives the Lorentz transformation properties of traceless part of the real symmetric gravitational metric tensor field $g_{\mu\nu}(x)$ with 10 components.

The 2-dimensional representations $(1/2, 0)$ and $(0, 1/2)$ are complex and the corresponding fields are called spinors. They transform like spin $1/2$ wave functions in quantum mechanics, but there do not appear to be any macroscopic classical fields with these properties.

Direct sums of the $(j_+, j_-)$ representations are also useful in classical and quantum theories. The sum $(1, 0) \oplus (0, 1)$ is real and represents the Lorentz transformation properties of the electromagnetic field $F_{\mu\nu}$. The bispinor representation $(1/2, 0) \oplus (0, 1/2)$ transforms like 4-component Dirac spinors of massive electrons and positrons.
3 Representations of the Poincaré Group

Classical fields are not naturally associated with elementary particles with quantized spin degrees of freedom. To make the connection it is necessary to quantize the fields. The $(j_+, j_-)$ representations of the Lorentz group can then be associated with the spin quantum numbers of elementary fermions and bosons.

The article by Wigner\cite{1} classifies all reducible representations of the Poincaré group by considering the little group of a quantum eigenstate with definite momentum $p^2 = E^2 - p^2 = m^2$.

The case $m^2 > 0$ corresponds to particles with non-zero mass with little group SU(2). In the rest frame of the particle $p = 0$, the irreducible representations are precisely the $s = 0, \frac{1}{2}, 1, \ldots$ spin angular momentum states of non-relativistic quantum mechanics.

The case $m^2 = 0$ is qualitatively different. A massless particle travels with the speed of light, and it is not possible to transform to its rest frame. Wigner showed that the little group is the 2-dimensional Euclidean group E(2) or ISO(2). This is essentially the group of rotations and reflections in a 2-dimensional plane perpendicular to the momentum $p$ of the particle state. The Lie group is abelian: there is only one rotation generator, which commutes with the reflections. The Lie algebra is trivial (no non-zero commutators). The irreducible representations of E(2) are all 1-dimensional.

A massless particle has a single polarization state that is invariant under proper orthochronous Lorentz transformations with integer or half-integer spin component. For example, a free electron neutrino has spin $-\frac{1}{2}$ in the direction of motion, and an anti-neutrino has spin $+\frac{1}{2}$. 
Photons have two polarization states with spin $\pm 1$ corresponding to right and left circular polarization. The two states are invariant under proper Lorentz transformations but mix under parity.

Gravitons are expected to be massless with two polarization states with spin angular momentum $\pm 2\hbar$. 
References

