Homework Assignment 12

Please format your submission as a single PDF file according to the instructions in the syllabus and submit it on UBlearns before 11:59 pm on Wednesday, December 18.

Problem 1: A zeroth-order natural relation: Peskin-Schroeder Problem 11.2. This problem studies an $N = 2$ linear sigma model coupled to fermions

$$
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi^j \right)^2 + \frac{1}{2} \left( \phi^j \right)^2 - \frac{\lambda}{4} \left( (\phi^j)^2 \right)^2 + \bar{\psi} (i \not{\partial}) \psi - g \bar{\psi} \left( \phi^1 + i \gamma^5 \phi^2 \right) \psi,
$$

where $\phi^j$ is a two component field, $j = 1, 2$.

(a) Show that this theory has the following global symmetry:

$$
\begin{align*}
\phi^1 &\to \cos \alpha \phi^1 - \sin \alpha \phi^2, \\
\phi^2 &\to \sin \alpha \phi^1 + \cos \alpha \phi^2, \\
\psi &\to e^{-i \alpha \gamma^5 / 2} \psi.
\end{align*}
$$

Show also that the solution to the classical equations of motion with the minimum energy breaks this symmetry spontaneously.

(b) Denote the vacuum expectation value of the field $\phi^j$ by $v$ and make the change of variables

$$
\phi^j(x) = (v + \sigma(x), \pi(x)).
$$

Write out the Lagrangian density in these new variables, and show that the fermion acquires a mass given by

$$
m_f = g \cdot v.
$$

Problem 2: The Gross-Neveu Model: Peskin-Schroeder Problem 11.3. The [Gross-Neveu model](#) is a model in two spacetime dimensions of fermions with a discrete chiral symmetry:

$$
\mathcal{L} = \bar{\psi}_j i \not{\partial} \psi_j + \frac{1}{2} g^2 \left( \bar{\psi}_j \psi_j \right)^2 \quad j = 1, 2, \ldots, N.
$$

The kinetic term of two-dimensional fermions is built from matrices $\gamma^\mu$ that satisfy the two-dimensional Dirac algebra. These matrices can be $2 \times 2$:

$$
\gamma^0 = \sigma^2 = \sigma_{y, \text{Pauli}}, \quad \gamma^1 = i \sigma^1 = i \sigma_{x, \text{Pauli}}.
$$

Define

$$
\gamma^5 = \gamma^0 \gamma^1 = \sigma^3 = \sigma_{z, \text{Pauli}},
$$

which anticommutes with the $\gamma^\mu$.

(a) Show that this theory is invariant with respect to

$$
\psi_j \to \gamma^5 \psi_j,
$$
and that this symmetry forbids the appearance of a fermion mass.

(b) Show that the functional integral for this theory can be represented in the following form:

\[ \int D\overline{\psi} D\psi e^{i \int d^2 x \mathcal{L}} = \int D\overline{\psi} D\psi D\sigma \exp \left[ i \int d^2 x \left\{ \overline{\psi} i \cdot \partial \psi j - \sigma \overline{\psi} j \psi j - \frac{1}{2g^2} \sigma^2 \right\} \right], \]

where \( \sigma(x) \) (not to be confused with a Pauli matrix) is a new scalar field with no kinetic energy terms.

(c) Compare the Gross-Neveu model with the Nambu - Jona-Lasinio model of dynamical symmetry breaking and explain the similarities and differences.

Problem 3: Frequency summations: Altland-Simons §4.5 page 185. Using the frequency summation techniques developed in the text, this problem involves the computation of two basic correlation functions central to the theory of the interacting Fermi gas.

(a) The pair correlation function \( \chi_{n,q}^c \) is an important building block entering the calculation of the Cooper pair propagator in superconductors. It is given by

\[ \chi_{n,q}^c \equiv -\frac{T}{L^d} \sum_{m,p} G_0(p, i\omega_m)G_0(-p + q, -i\omega_m + i\omega_n) = \frac{1}{L^d} \sum_p \frac{1 - n_F(\xi_p) - n_F(\xi_{p+q})}{i\omega_n - \xi_p - \xi_{p+q}}, \]

where

\[ G_0(p, i\omega_m) = \frac{1}{i\omega_m - \xi_p}. \]

Verify the second equality. (Note that \( \omega_m = (2m + 1)\pi\beta \) are fermionic Matsubara frequencies, while \( \omega_n = 2\pi n T \) is a bosonic Matsubara frequency.)

(b) Another correlation function central to the theory of the interacting Fermi gas (see Section 5.2), the so-called density-density response function, is given by

\[ \chi_{q,\omega_n}^d \equiv -\frac{T}{L^d} \sum_{m,p} G_0(p, i\omega_m)G_0(p + q, i\omega_m + i\omega_n) = -\frac{1}{L^d} \sum_p \frac{n_F(\xi_p) - n_F(\xi_{p+q})}{i\omega_n + \xi_p - \xi_{p+q}}. \]

Again verify the second equality.

Problem 4: Fluctuation contribution to the Ginzburg-Landau action of the superconductor: Altland-Simons §6.7 page 334. In this short problem we derive the energy cost corresponding to large-scale spatial fluctuations of the order parameter of a BCS superconductor.

(a) Consider the second-order contribution to the Ginzburg-Landau action of the BCS superconductor Eq. (6.31)

\[ S^{(2)}[\Delta, \bar{\Delta}] = \sum_q \Gamma_q^{-1} |\Delta(q)|^2, \quad \Gamma_q^{-1} = \frac{1}{g} - \frac{T}{L^d} \sum_p G_p G_{-p+q}. \]

In Problem (1) above you have evaluated the frequency summation involved in the definition of the integral kernel \( \chi_{n,q}^c \). Expand \( \chi_{0,q}^c \) to second order in \( q \). Hints: You may trade the momentum summation for an integral, and linearize the dispersion: \( \xi_{p+q} \simeq \xi_p + p \cdot q/m \) (think why is this a permissible simplification). You can also use the identity

\[ \int d\epsilon \frac{1}{\epsilon} \partial_\epsilon n_F(\epsilon) = \frac{e}{T^2}, \quad e = \frac{7\zeta(3)}{2\pi^2}, \quad \text{and} \quad \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}. \]
is the Riemann zeta function.

**Problem 5: A model with two Higgs fields:** Peskin-Schroeder Problem 20.5.

(a) Consider a model with two scalar fields $\phi_1$ and $\phi_2$, which transform as SU(2) doublets with $Y = \frac{1}{2}$. Assume that the two fields acquire parallel vacuum expectation values of the form

$$
\langle \phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}.
$$

Show that these vacuum expectation values produce the gauge boson masses as in the Standard Model Higgs field with the replacement

$$v^2 \rightarrow (v_1^2 + v_2^2).$$

(b) The most general potential function for a model with two Higgs doublets is quite complex. However, if we impose the discrete symmetry $\phi_1 \rightarrow -\phi_1$, $\phi_2 \rightarrow -\phi_2$, the most general potential is

$$V(\phi_1, \phi_2) = -\mu_1^2 \phi_1^\dagger \phi_1 - \mu_2^2 \phi_2^\dagger \phi_2 + \lambda_1 (\phi_1^\dagger \phi_1)^2 + \lambda_2 (\phi_2^\dagger \phi_2)^2 + \lambda_3 (\phi_1^\dagger \phi_1) (\phi_2^\dagger \phi_2) + \lambda_4 (\phi_1^\dagger \phi_2) (\phi_2^\dagger \phi_1) + \lambda_5 (\phi_1^\dagger \phi_2)^2 \text{ h.c.}.$$ 

Find conditions on the parameters $\mu_i$ and $\lambda_i$, so that the configuration of vacuum expectation values required in part (a) is a locally stable minimum of this potential.