Chapter 5: Straight Level Flight - Jet Aircraft

Mohammad Sadraey
Daniel Webster College

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Chapter 5

Straight Level Flight - Jet Aircraft

5.1. Introduction

Each regular flight operation usually includes take-off, climb, cruise, turn, descend, and landing (figure 5.1). A civil aircraft is in cruising flight for much of the duration of its flight. Cruising flight is a major part of an air mission and is defined as a straight line flight that the velocity and altitude are often kept almost constant. In other word, there will be no climbing and descending in cruise. In this book, we begin the performance analysis of different flight phases from the simplest one, i.e., straight line level flight.

This chapter and the next chapter devoted to the investigation of various flight parameters of an aircraft when it is in cruising flight. In chapter four, we classified aircraft engines into two main groups; one propeller-driven engines (piston prop and turboprop), and another one jet engines (turbojet and turbofan). The main difference is that the output of a jet engine in thrust (T), while the output of a prop engine is power (P), but is converted to thrust by its propeller.

Because of this difference, the derivations of basic relationships and applied equations for these two groups of aircraft are independent and the results are considerably dissimilar. Therefore the analysis of aircraft performance in cruising flight is presented in two chapters. The performance of aircraft with turbojet and turbofan engines in straight-line level flight is introduced in this chapter. While, in chapter six, the performance of aircraft with piston prop and turboprop engines in straight-line level flight is presented.

This chapter is of critical importance, since the conceptual design of major components of powered aircraft such as wing and horizontal tail are based on satisfying the requirements in
this flight phase. On the other hand, the easiest flight operation is the cruising flight such that the automatic flight control of civil transport airplane is realizable and currently operational in most transport aircraft. If the reader does not capture the fundamentals of cruising flight, the success in understanding the other flight phases would not be easy.

Figure 5.1. The flight phases of a simple flight mission

In this chapter, three assumptions are made in subsonic flight regime to make the derivations of equations easier.

a. The zero lift drag coefficient \((C_{Do})\) is assumed as constant during subsonic flight.

b. Drag force \((D)\) variations versus velocity \((V)\) is fitted with a parabolic curve.

c. The induced drag correction factor; \(K\), in the drag polar \((C_D = C_{Do} + K C_{L^2})\) is assumed constant in subsonic flight.

If the reader is looking for a more precise analysis, (s)he must derive performance equations without the above assumptions. In majority of the cases, these assumptions do not considerably degrade the results of the analysis.

This chapter is organized as follows: First, basic equations in straight-line level flight are introduced and derived. Then the methods to evaluate the following performance criteria are discussed: 1. specific speeds in straight line level flight; 2. Range; 3. Endurance; 4. Ceiling.

The specific speeds in straight line level flight include maximum speed, cruising speed, minimum drag speed, maximum range speed, speed for absolute ceiling, and maximum endurance speed. It is emphasized again that these topics are discussed for jet aircraft, and in the next chapter they are re-introduced for prop aircraft. In this chapter, several statistical tables have been prepared to illustrate the real data of current aircraft performance specifications. They provide the reader with a feeling of how a given aircraft is performing. For the purpose of simplicity, we sometimes use "cruising flight" instead of "straight line level flight". Although, velocity is a "vector" quantity and speed is a "scalar" value, in this book, we use both velocity and speed interchangeably, but we mean a vector quantity in both cases. In majority of the cases, when we use the term "speed" or "velocity", in fact we mean aircraft speed or "airspeed".

5.2. Fundamental Equations

This section deals with the fundamental equations and basic parameters that are employed in various performance specifications of cruising flight. They are derived here and then will be
applied in later sections to derive various relationships to evaluate cruising flight performance. The equations and parameter are: steady-state trim equations, relationship between drag and thrust with speed, relationship between speed and angle of attack, and maximum lift to drag ratio.

5.2.1. Steady-State Trim Equations

The most fundamental equations of motion in cruising flight are based on the Newton’s second law. The second law of Newton states that when a net external force \( \Sigma F \) acts on an object of mass \( m \), the linear acceleration \( \mathbf{a} \) that results is directly proportional to the net force and has a magnitude that is inversely proportional to the mass. The direction of the acceleration is the same as the direction of the net applied force. In equation form, we have:

\[
\sum F = ma \tag{5.1}
\]

![Figure 5.2. Aircraft coordinate system](image)

The external forces that include aircraft weight (W); engine thrust (T), drag (D), and lift (L) are derived in chapters two, three and four. An aircraft can move along three axis, as we call them x, y, and z (figure 5.2). For simplicity, we assume that x axis coincides with aircraft body (fuselage center line) axis. Both the forces and the resulting accelerations are vector quantity, so net force in Newton’s second law has components \( \Sigma F_x, \Sigma F_y, \Sigma F_z \), while the acceleration \( \mathbf{a} \) has components \( a_x, a_y, \) and \( a_z \). Consequently, Newton’s second law can be written in an equivalent form as three equations, one for x component, one for y component, and one for z component. In cruising flight, the altitude is assumed to be constant, as well as the direction, thus there will be no acceleration in y and z directions. Therefore Newton’s second law is expressed in the following three equations:

\[
\begin{align*}
\sum F_x &= ma_x \tag{5.2a} \\
\sum F_y &= 0 \tag{5.2b} \\
\sum F_z &= 0 \tag{5.2c}
\end{align*}
\]

In cruising flight, we do not consider any turning motion and side force, so we ignore the y-component equations of motion, and then the equations are reduced to two equations:

\[
\sum F_x = ma_x \tag{5.3a} \\
\sum F_z = 0 \tag{5.3b}
\]

The equations 5.3 demonstrate an accelerated flight, and it means that the flight speed is increasing or decreasing. The straight level flight is often un-accelerated (i.e., constant cruising speed). In such a case, the equations are simplified as follow:
\[ \sum F_x = 0 \quad (5.4a) \]
\[ \sum F_z = 0 \quad (5.4b) \]

These equations demonstrate the force equations of motion. The moment equations of motion are also part of the equations of motion, but they are mainly used in analyzing the stability and control. So, they are not considered in this chapter. With respect to four forces shown in figure 5.3, the basic equations of motion that governs the straight line flight are as follows:

\[ T = D \quad (5.5) \]
\[ W = L \quad (5.6) \]

Figure 5.3. Forces on an aircraft in straight line level flight

According to these equations, two pre-requisites for un-accelerated straight-line flight are as follows: the drag force must be equal to the engine thrust, and the lift force must be equal to the weight of the aircraft. In another word, the engine must produce enough thrust to counteract the drag force, and the aircraft (mainly wing) must generate enough lift force to hold the aircraft against the weight. This implies that the aircraft is in trim or equilibrium state. An aircraft has a low angle of attack (less than 5 degrees) during cruising flight and has low engine setting angle. Figure 5.4 shows a cruising aircraft with an angle of attack and engine setting angle. Thus the trim equation in cruise will be:

\[ D = T \cos(\alpha + i_e) \quad (5.7) \]
\[ W = L + T \sin(\alpha + i_e) \quad (5.8) \]
where $\alpha$ is the aircraft angle of attack and $i_e$ is the engine setting angle. More details will be presented in Section 5.2.3.

**Example 5.1**
A jet transport aircraft with a mass of 120,000 kg is cruising with a speed of 400 knot at sea level. If the drag force is 290,000 N, the aircraft angle of attack is 3 degrees and the engine setting angle is 4 degrees,

a. how much thrust the engine is producing?,
b. how much lift force the aircraft is providing?

**Solution:**

$$D = T \cos(\alpha + i_e) \Rightarrow T = \frac{D}{\cos(\alpha + i_e)} = \frac{290,000}{\cos(3 + 4)} \Rightarrow T = 292177.8 \; N \quad (5.7)$$

$$W = L + T \sin(\alpha + i_e) \quad (5.8)$$

$$\Rightarrow L = W - T \sin(\alpha + i_e) = (120,000 \times 9.81) - (292177.8 \sin(3 + 4)) \Rightarrow L = 1141592.5 \; N$$

**5.2.2. Drag, Thrust and Velocity Relationship**

When an aircraft is flying in a straight line at a constant altitude, if the pilot applies more throttle setting, the thrust will be increased, and then the aircraft speed will be increased accordingly. In addition, as the velocity is increased, the drag is increased too. The velocity is increased until the drag and thrust gets equal. So, for each value of aircraft weight and engine thrust, there is usually one airspeed (often two) which satisfies trim condition. The flight in the first and final states are called unaccelerated and the flight in between is called accelerated flight.

The variations of aircraft drag and engine thrust as functions of velocity at a particular altitude are shown in Figure 5.5. It is assumed that the engine thrust is independent of aircraft velocity. As we expect, the aircraft drag is a nonlinear function of velocity (parabolic). The drag curve is a parabola and has a minimum value. Four velocities in this figure are worth noting. In fact, the thrust is a required thrust, but the drag is a produced drag in any particular speed. In other word, the thrust is the producer of the velocity, but the velocity is the reason of aircraft drag. The following conclusions could be made by considering equation 5.7 and figure 5.5:

- There are low engine thrusts such as $T_1$ that they are not enough for cruising flight. Since there is no intersection between some thrust lines and the drag curve, the aircraft is not able to have cruising flight at such thrust values.
- At any thrust that is less than maximum and more than $T_2$, the thrust line has two intersections with drag curve. This means, the aircraft may have two cruising velocities at such particular thrust. It other word, there are two different cruising velocities that the aircraft is producing the same drag. The reason for this is that, there is a huge increase in induced drag at low velocities due to high lift coefficient.
- The thrust $T_2$ is a minimum thrust that cruising flight is possible. This velocity in referred to as minimum drag velocity ($V_{minD}$). In this case, there is only one intersection between drag curve and thrust lines.
- If the pilot employs the maximum engine thrust ($T_{max}$) at any particular altitude, the aircraft will have its maximum velocity at that particular altitude.
Example 5.2

An aircraft with a mass of 2,500 kg, a wing area of 18 m$^2$ and a drag coefficient of 0.04 is cruising with a speed of 160 knot at constant altitude. If the pilot increases the engine thrust by 20%, calculate the initial acceleration and the final velocity. You may assume the drag coefficient is constant throughout this flight and the aircraft is flying at ISA condition and sea level.

Solution:

In a cruising flight with constant altitude, thrust is equal to drag, while is the drag is equal to:

$$D = \frac{1}{2} \rho V^2 SC_D$$

Thus

$$T_i = D_i = \frac{1}{2} \rho V_i^2 SC_D = \frac{1}{2} \times 1.225 \times (160 \times 0.5144)^2 \times 18 \times 0.04 \Rightarrow T_i = 2987.3 \text{ N}$$

When the engine thrust is increased by 20%:

$$T_2 - D_i = ma \Rightarrow a = \frac{T - D}{m} = \frac{(1.2 \times 2987.3) - 2987.3}{2500} \Rightarrow a = 0.239 \frac{m}{\text{sec}^2}$$

This acceleration will increase the aircraft velocity until the drag is equivalent to thrust.

$$T_2 = D_2 \Rightarrow 1.2T_i = \frac{1}{2} \rho V_2^2 SC_D$$

$$\Rightarrow V_2 = \sqrt{\frac{1.2T_i}{\frac{1}{2} \rho SC_D}} = \sqrt{\frac{1.2 \times 2987.3}{0.5 \times 1.225 \times 18 \times 0.04}} = 90.16 \frac{m}{\text{sec}} = 175.27 \text{ knot}$$

The ratio between the new speed and the previous speed is:
\[
\frac{V_2}{V_1} = \frac{175.27}{160} = 1.1
\]
It means that the aircraft velocity is increased 10%, although the thrust is increased by 20%.

5.2.3. Velocity-Angle-of-Attack Relationship
The straight-line sustained level flight is possible at any permissible velocity. A permissible velocity is defined as any velocity equal or higher than the stall speed and equal or less than the maximum speed. For an aircraft to stay in the air, it must provide enough lift through its aerodynamic components (mainly by wing). The aircraft cannot produce a sufficient lift force if the velocity is less than stall speed. The aircraft also is not able to generate sufficient thrust to have velocity higher than its maximum velocity. If a pilot needs to maintain the altitude (that air density is constant), (s)he must maintain the lift force equal to the weight of the aircraft. The lift force is a function of air density (\(\rho\)), lift coefficient, wing area, and airspeed:

\[
L = \frac{1}{2} \rho V^2 S C_L
\]  
(2.3)
This equation implies that in order to change the airspeed in a cruising flight with a constant aircraft weight, the lift coefficient (\(C_L\)) should be inversely varied. To increase the velocity, the lift coefficient (\(C_L\)) needs to be decreased, and to reduce the velocity, the lift coefficient (\(C_L\)) needs to be increased. To vary the lift coefficient in a constant altitude, the aircraft angle of attack; or flap deflection; must be changed. Thus lowering the angle of attack (\(\alpha\)), increases the aircraft velocity, and increasing the angle of attack (\(\alpha\)), reduces the aircraft velocity. However, in a long cruising flight, since the weight of aircraft is constantly decreasing (due to burning fuel), the angle of attack (\(\alpha\)) must be reduced accordingly. This is necessary to maintain constant velocity and constant altitude.

![Diagram](image)

Figure 5.6. One aircraft in two different cruising flight conditions
Figure 5.6 demonstrates an aircraft in two cruising flight conditions: 1. Three-degree angle of attack, 2. Six-degree angle of attack. In both conditions the aircraft has constant altitude, but different velocity (220 knot and of 120 knot). In figure 5.6a the aircraft has three degrees of \(\alpha\) (the corresponding \(C_L\) is 0.5), since it cruises with the velocity of 220 knot. In figure 5.6b the
Aircraft has six degrees of $\alpha$ (the corresponding $C_L$ is 0.8), since it cruises with a lower velocity of 120 knot. Thus, a higher velocity ($V$) requires a lower lift coefficient ($C_L$) and subsequently a lower angle of attack ($\alpha$). The aircraft angle of attack is measured with respect to its zero-lift line. However, the wing angle of attack is measured with respect to a flat reference such as the cabin/cockpit floor.

Figure 5.7 demonstrates a typical relationship between the aircraft airspeed and the angle of attack for a cruising flight of an aircraft with a mass of 5,100 kg and a reference wing area of 25 m$^2$. In every modern fixed-wing aircraft, the wing setting angle is fixed; therefore a change in aircraft angle of attack implies a change in wing angle of attack. This figure illustrates that as the aircraft airspeed is increased, both lift coefficient and angle of attack are decreased. For instance, the cruising angle of attack of a supersonic aircraft such as Concorde is about 0.3-1 degrees, while the cruising angle of attack of a high subsonic aircraft such as Boeing 747 is about 3-5 degrees.

![Figure 5.7. Typical variations of lift coefficient and angle of attack versus velocity in a cruising flight](image)

The maximum allowable angle of attack ($\alpha_{\text{max}}$) that is often referred to as the stall angle ($\alpha_s$) determines the maximum lift coefficient ($C_{L\text{max}}$). This maximum lift coefficient ($C_L$) in turn determines the minimum permissible velocity for a cruising flight. This velocity is referred to as stall speed ($V_s$). For the aircraft in figure 5.7, the stall angle is nine degrees and the stall speed is 60 m/sec. Figure 5.8 compares the angle of attacks of two fast supersonic aircraft (General Dynamics (now Lockheed Martin) F-16 Fighting Falcon and Aérospatiale-BAC supersonic transport aircraft Concorde) with two slow subsonic aircraft, when they are flying at the same
speed. As you see, the angles of attack of supersonic aircraft are much higher than those of subsonic aircraft, since they are designed to cruise supersonically.

The wing is the main component (i.e., major contributor) to produce lift force for the entire aircraft. Therefore, the criterion for maximum permissible aircraft angle of attack (or aircraft stall angle) is the wing stall angle. The aircraft angle of attack ($\alpha$) is a function of wing angle of attack ($\alpha_w$), fuselage angle of attack ($\alpha_f$) and horizontal tail angle of attack ($\alpha_{ht}$). The wing is in turn attached to the fuselage with an angle called wing setting angle ($i_w$). Thus the wing angle of attack is in fact the wing setting angle plus fuselage angle of attack:

$$\alpha_w = \alpha + i_w$$  \hspace{1cm} (5.9)

![Figure 5.8. Comparison between angle of attacks of a high speed (Concorde (left) and F-16 (right)) and low speed aircraft (BAE Hawk (left) and P-51 Mustang (right))](image)

In other word, the pilot must always be mindful that the wing angle of attack does not reach/pass its stall angle ($\alpha_s$).

### 5.2.4. Maximum Lift-to-Drag Ratio (L/D)$_{\text{max}}$

One of the important parameters in a cruising flight is lift-to-drag ratio. One of the objectives in aerodynamic design of an aircraft is to design an aircraft which produces the maximum lift with a minimum drag. In this regard, lift-to-drag ratio is playing a significant role in the aircraft design process. This design objective has several applications. If we divide equation 5.5 to equation 5.6, we will have:

$$\frac{L}{D} = \frac{W}{T} \Rightarrow T = \frac{W}{(D/L)} = \frac{W}{(L/D)}$$  \hspace{1cm} (5.10)

Mathematically, in order to minimize a ratio with a constant numerator, the denominator should be maximized. Recall that the minimum value of D/L is the reciprocal of the maximum value of L/D.

Hence

$$T_{\text{min}} = \frac{W}{(L/D)_{\text{max}}}$$  \hspace{1cm} (5.11)

This relationship states that if a pilot intends to have a minimum fuel consumption (i.e., minimum thrust), (s)he must fly with a configuration that has a maximum lift-to-drag ratio. This configuration for any aircraft is unique, so must already be calculated and known. Now let’s see how we can determine the flight configuration for this purpose.
In a cruising flight, lift is equal to the aircraft weight. So, in order to maximize the lift-to-drag ratio; the drag should be minimized; 
\[
(L/D)_{\text{max}} \rightarrow D_{\text{max}}
\]  
(5.12)

In effect, the question for maximizing the lift-to-drag ratio; yields a question for minimizing the drag. At the beginning, it may seem that we have to have a flight with a zero angle of attack, in order to minimize aerodynamic drag of the airplane. This perception is incorrect, because the reduction of angle of attack means an increase the aircraft velocity, and this requires more thrust. Then, it may seem that the solution is to reduce the flight speed. This is incorrect either, because a reduction in the airspeed necessitates an increase in the angle of attack, and this often causes more drag force. Hence we have to solve this problem analytically. We begin our analysis with the definitions of lift and drag:

\[
L = \frac{1}{2} \rho V^2 SC_L
\]
(2.3)

\[
D = \frac{1}{2} \rho V^2 SC_D
\]
(2.4)

The division of the lift over the drag yields:

\[
\frac{L}{D} = \frac{\frac{1}{2} \rho V^2 SC_L}{\frac{1}{2} \rho V^2 SC_D} = \frac{C_L}{C_D}
\]
(5.13)

By inspection, we can readily conclude the following:

\[
\left( \frac{L}{D} \right)_{\text{max}} = \left( \frac{C_L}{C_D} \right)_{\text{max}}
\]
(5.14)

Therefore, in order to maximize the lift-to-drag ratio, one should maximize the ratio of lift coefficient to the drag coefficient. In chapter three, the drag polar was defined as

\[
C_D = C_{D_0} + KC_L^2
\]
(5.15)

This can be plugged into the ratio of lift coefficient to drag coefficient

\[
\frac{C_L}{C_D} = \frac{C_L}{C_{D_0} + KC_L^2}
\]
(5.16)

The aircraft lift-to-drag ratio is a function of lift coefficient. To find maximum C_L/C_D, one must differentiate equ 5.16 with respect to lift coefficient, and set the result equal to zero (i.e., zero slope (Figure 5.10)):

\[
\frac{d}{dC_L} \left( \frac{C_L}{C_D} \right) = \frac{C_{D_0} + KC_L^2 - 2KC_L^3}{(C_{D_0} + KC_L^2)^2} = 0
\]
(5.17)

If we set the right hand side equal to zero, we obtain:

\[
C_{D_0} + KC_L^2 - 2KC_L^3 = 0
\]
(5.18)

or

\[
C_{D_0} = KC_L^2
\]
(5.19)

Substitution of the equation 5.19 into equation 5.15 yields:

\[
C_{D_{\text{Lmax}}} = 2C_{D_0}
\]
(5.20)

This implies that, when the lift-to-drag ratio is at its maximum value, the drag coefficient will be twice the zero lift drag coefficient. In other word, the zero lift drag coefficient will be equal to the induced drag coefficient.
Every aircraft has a unique maximum lift-to-drag ratio. Table 5.2 demonstrates the maximum lift to drag ratio for several airplane categories.

To calculate this parameter, the equation 5.19 is inserted into equation 5.16:

\[
\left( \frac{C_L}{C_D} \right)_{\text{max}} = \frac{C_L}{KC_L^2 + KC_L^2 + 2KC_L^2} = \frac{1}{2KC_L^2} = \frac{1}{2KC_{I(L,0)\text{max}}} 
\]

Furthermore, when the lift-to-drag ratio is at its maximum value, from equation 5.19, we obtain:

\[
C_{l(L,0)\text{max}} = \sqrt{\frac{C_{D_0}}{K}}
\]

When this lift coefficient is plugged into equation 5.17, the following is obtained

\[
\left( \frac{C_L}{C_D} \right)_{\text{max}} = \frac{1}{2\sqrt{KC_{D_0}}}
\]

With this relationship, one is able to evaluate the maximum lift-to-drag ratio of any aircraft. The only necessary information is the aircraft zero lift drag coefficient (\(C_{D_0}\)) and the induced drag correction factor (K).

The equation 5.23 is a mathematical expression; the theoretical value of \(C_{l(L,0)\text{max}}\) must be within a practical flight limit. The value of \(C_{l(L,0)\text{max}}\) from equation 5.23 cannot be more than the aircraft maximum lift coefficient (\(C_{l\text{max}}\)). If the output of the equation is more than \(C_{l\text{max}}\), ignore the outcome, and select a new value slightly less than \(C_{l\text{max}}\).

**Example 5.3**

The aircraft Cessna Citation II has the following features:

\[C_{D_0} = 0.022, e = 0.85, AR = 8.3, m = 6,032 \text{ kg}, S = 30 \text{ m}^2, a = 5.9 \text{ 1/rad}, V_s = 82 \text{ knot}, T_{\text{max}} = 22,240 \text{ N}\]

Determine the followings:

a. Maximum lift-to-drag ratio
b. What is the angle of attack corresponding to the maximum lift-to-drag ratio (at sea level)? Assume \(\alpha_o = 0\).
c. What is the airspeed corresponding to the maximum lift-to-drag ratio (at sea level)?
d. What is the minimum required thrust for this aircraft to fly?

**Solution:**

\[K = \frac{1}{\pi \cdot e \cdot AR} = \frac{1}{3.14 \times 0.85 \times 8.3} \Rightarrow K = 0.045 \]
\[
\left(\frac{C_L}{C_D}\right)_{\text{max}} = \frac{1}{2\sqrt{KC_D}} = \frac{1}{2\sqrt{0.045 \times 0.022}} \Rightarrow \left(\frac{C_L}{C_D}\right)_{\text{max}} = 15.89 \quad (5.24)
\]

b.

\[
C_L = \sqrt{\frac{C_D}{K}} = \sqrt{\frac{0.022}{0.045}} \Rightarrow C_L = 0.69 \quad (5.23)
\]

\[
a = \frac{dC_L}{d\alpha} \Rightarrow 5.9 = \frac{0.69}{\alpha} \Rightarrow \alpha = 0.11 \text{ rad} = 6.7 \text{ deg} \quad (2.12)
\]

c.

\[
W = L = \frac{1}{2} \rho V^2 S C_L \Rightarrow V = \sqrt{\frac{2W}{\rho S C_L}} = \sqrt{\frac{2 \times 6032 \times 9.81}{1.225 \times 30 \times 0.69}} \quad (2.47)
\]

\[
\Rightarrow V = 68.3 \text{ m/sec} = 132.82 \text{ knot}
\]

This speed is about two times the stall speed.

d.

\[
C_D = 2C_{D_{\text{a}}} = 2 \times 0.022 = 0.044 \quad (5.20)
\]

\[
T = D \quad (5.5)
\]

\[
D = \frac{1}{2} \rho V^2 S C_D = 0.5 \times 1.225 \times (68.3)^2 \times 30 \times 0.044 \Rightarrow T_{\text{min}} = 3771.5 \text{ N} \quad (3.1)
\]

This implies that the minimum thrust to fly is about 17% of maximum thrust.

\[
\frac{T_{\text{min}}}{T_{\text{max}}} = \frac{3771.5}{22,240} = 0.17
\]

This example demonstrates that if the Cessna Citation II needs to cruise with the minimum thrust at sea level, the airspeed will be 132.8 knot; and that the angle of attack would be 6.7 degrees. In this flight condition, propulsion system employs only 17% of its thrust. The lift-to-drag ratio will be at its maximum value, which is 15.89.

Figure 5.9 shows the generic variations of airfoil C\textsubscript{l}/C\textsubscript{d} with respect to the angle of attack for an airfoil. The variations curve of airfoil lift-to-drag ratio versus the angle of attack has a similar shape. The variations of drag, and lift-to-drag ratio versus speed for a jet aircraft with a mass of 5,000 kg is demonstrated in Table 5.1. This graphical representation of this table is sketched in figure 5.10. It is noticed that the lift-to-drag ratio reaches a maximum value of 12. In this flight condition, the drag force is at its minimum value (i.e., 4,170 N). This aircraft will have the maximum lift-to-drag ratio, if it cruises with a 4 degrees of angle of attack. In general, any aircraft has one value for its maximum lift-to-drag ratio. This parameter is usually similar for a group of aircraft with similar configuration.

5.3. Specific Speeds in Straight Line Level Flight

An aircraft is able to have a sustained cruising flight with various airspeeds from a minimum speed to a maximum speed. The minimum speed is usually little more than the stall speed (about 1.1 \(V_{\text{s}}\) to 1.3 \(V_{\text{s}}\)). The level flight at stall speed is challenging (sometimes dangerous), but cruising flight with a velocity less than the stall speed is impossible. The reason is that the lift force would be less than weight, and hence holding altitude at this situation is not realizable.
The maximum speed in cruising flight is when the pilot uses the maximum engine throttle. The minimum and maximum speeds of an aircraft depend on aircraft weight, flap setting, and flight altitude.

![Graph](image)

**Figure 5.9. Typical variations of C_l/C_d with respect to angle of attack for an airfoil**

![Table](image)

**Table 5.1. The variations of drag, angle of attack, and lift-to-drag ratio for a jet aircraft**

For a GA aircraft, the maximum speed is often about 2-3 times the stall speed. This ratio for jet transport aircraft is about 4 to 5; and for a jet fighter is about 5 to 15. The specific speeds are of significant importance for pilots and are used at specific flight conditions. A few examples are:
- maximum speed ($V_{\text{max}}$)
- minimum drag speed ($V_{\text{minD}}$)
- speed for maximum lift-to-drag ratio ($V_{\text{(L/D)max}}$)
- maximum range speed ($V_{\text{maxR}}$)
- maximum endurance speed ($V_{\text{maxE}}$)
- speed for absolute ceiling
- cruise speed ($V_c$)

![Figure 5.10. The variations of drag and lift-to-drag ratio versus speed for a jet aircraft](image)

**Table 5.2. The typical maximum lift-to-drag ratio for various aircraft**

<table>
<thead>
<tr>
<th>No</th>
<th>Aircraft type</th>
<th>(L/D)$_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Sailplane (glider)</td>
<td>30-40</td>
</tr>
<tr>
<td>2</td>
<td>Jet transport</td>
<td>15-20</td>
</tr>
<tr>
<td>3</td>
<td>Light general aviation</td>
<td>10-15</td>
</tr>
<tr>
<td>4</td>
<td>Subsonic fighter</td>
<td>7-10</td>
</tr>
<tr>
<td>5</td>
<td>Supersonic fighter</td>
<td>4-7</td>
</tr>
<tr>
<td>6</td>
<td>helicopter</td>
<td>3-5</td>
</tr>
<tr>
<td>7</td>
<td>Remote controlled</td>
<td>7-16</td>
</tr>
<tr>
<td>8</td>
<td>Home-built</td>
<td>6-14</td>
</tr>
</tbody>
</table>

Every single speed of this list has one or more specific applications; and a pilot adopts one of them based on the desired mission. For instance, a jet transport aircraft is supposed to carry a payload with a minimum cost, while a fighter jet has the mission to fight with enemy fighter, or a surveillance aircraft has to fly over borders to protect it from illegal trespassing. Each aircraft must fly with a specific speed to carry its mission efficiently. Hence, knowing these speeds is a prerequisite of an economic or successful flight for a pilot. Figure 5.11 illustrates a civil transport aircraft Boeing 757 with two turbofan engines at cruising flight. Note that the angle of attack is considerable by inspection.

In this section, three specific speeds of maximum speed ($V_{\text{max}}$), minimum drag speed ($V_{\text{minD}}$), and speed for maximum lift-to-drag ratio ($V_{(L/D)_{\text{max}}}$) are analyzed. Four specific speeds of maximum range speed ($V_{\text{maxR}}$), maximum endurance speed ($V_{\text{maxE}}$), cruise speed ($V_c$), and speed for absolute ceiling are discussed in the sections 5.4 5.5 and 5.6 respectively. It is worth mentioning that velocity is a vector quantity and has magnitude plus direction, but speed is a scalar quantity and has only magnitude. The word “speed” is used in most cases just for easier application.
5.3.1. Maximum Speed ($V_{\text{max}}$)

One of the most important performance criteria for any aircraft is its maximum speed. The maximum sustained speed in level flight is achieved when the maximum engine thrust is employed. In order for the engine to have a longer life, and a lower fuel consumption, it is highly recommended not to fly with the maximum thrust in a long flight. Thus, an aircraft is rarely flown with its maximum speed. However, it is crucial for a fighter jet to succeed/survive in a battle field by flying with its maximum speed. When comparing two aircraft, one with a higher maximum speed is pronounced to have a better cruise performance.

Every year, the record of maximum speed is registered when it is increased. The experimental aircraft X-15A has achieved the highest maximum speed of 7,297 km/hr of Mach of 6.72 on October 3rd 1967. It is noticeable that this speed was obtained while X-15A was launched from another mother aircraft. The record of highest maximum speed for an independent aircraft flight in 1970’s belonged to Lockheed SR-71 Blackbird (Figure 5.13) that achieved a velocity of 1905.8 knot (little more than Mach 3). The current record for the maximum speed belongs to X-51 Wave Rider. The Boeing X-51 is an unmanned demonstration aircraft with a scramjet engine completed a flight of over six minutes and reached speed of over Mach 5 for 210 seconds on 1 May 2013 for the longest duration hypersonic flight.

The maximum speed of an aircraft depends on several parameters including the engine thrust, aircraft weight, and the cruising altitude. As the weight of aircraft (e.g., passenger, luggage, cargo, store, or fuel) is decreased, the maximum speed will be increased. Due to this fact, the published value of the maximum airspeed for an aircraft (e.g., by Jane’s [3]) is often based of the maximum aircraft weight and the altitude indicated.

Increase in the altitude has several impacts on flight. First, the air density is decreased with altitude, and because of this reduction, the aerodynamic forces of lift, and drag, and the engine thrust all are decreased. The reduction of these three forces is not at the same rate. To compensate for the lift reduction at high altitude, the lift coefficient (in effect, the angle of attack) must be increased. But an increase in the lift coefficient will induce an increase in drag coefficient. Simultaneously, the reduction of air density (with altitude) causes a drag reduction. This process has dual influence, such that at low altitude, maximum speed is increased, but at high altitude, causes a reduction in the maximum velocity. Thus, there is usually a specific altitude that maximum speed reaches it absolute maximum ($V_{\text{max max}}$). This altitude is one of the favorite altitudes to fly, when the pilot is going to absolutely maximize its maximum speed. Therefore, the maximum airspeed is a nonlinear function of the altitude.
Now let’s see how to analyze and calculate this absolute maximum speed and its corresponding altitude. In a cruising flight with a constant velocity, the drag force and thrust must be equal (equation 2.7). This can be applied for a flight with the maximum speed as follows:

\[ T_{\text{max}} = D_{\text{max}} = \frac{1}{2} \rho V_{\text{max}}^2 SC_D \]  
\[ \text{(5.25)} \]

In chapter two and three, we already determined the lift coefficient and drag polar as

\[ C_L = \frac{2W}{\rho V_{\text{max}}^2 S} \]  
\[ \text{(2.6)} \]

\[ C_D = C_{D_0} + KC_L^2 \]  
\[ \text{(3.10)} \]

By substituting equations 2.6 and 3.10 into equation 5.25, we obtain

\[ T_{\text{max}} = \frac{1}{2} \rho V_{\text{max}}^2 SC_{D_0} + \frac{2KW^2}{\rho V_{\text{max}}^2 S} \]  
\[ \text{(5.26)} \]

or

\[ \frac{1}{2} \rho V_{\text{max}}^2 SC_{D_0} + \left( \frac{2KW^2}{\rho S} \right) \frac{1}{V_{\text{max}}^2} - T_{\text{max}} = 0 \]  
\[ \text{(5.27)} \]

In chapter four, the jet engine thrust is introduced as a function of altitude (air density) as follows:

\[ T_{\text{max}} = T_{\text{max SL}} \left( \frac{\rho}{\rho_o} \right)^{0.9} \]  
\[ \text{(Turbojet; Troposphere)} \]  
\[ \text{(4.21)} \]

\[ T_{\text{max}} = T_{\text{max SL}} \left( \frac{\rho_{11000}}{\rho_o} \right)^{0.9} \left( \frac{\rho}{\rho_{11000}} \right) \]  
\[ \text{(Turbojet; Stratosphere)} \]  
\[ \text{(4.22)} \]

\[ T_{\text{max}} = T_{\text{max SL}} \left( \frac{\rho}{\rho_o} \right)^{1.2} \]  
\[ \text{(Turbofan engine)} \]  
\[ \text{(4.24)} \]

In these expressions, \( T_{\text{max}} \) and \( T_{\text{max SL}} \) are maximum thrust at any altitude and maximum thrust at sea level respectively.

By substituting these equations into equation 5.27, we will have the following nonlinear expression:

\[ AV_{\text{max}}^2 + \frac{B}{V_{\text{max}}^2} - CT_{\text{max}} = 0 \]  
\[ \text{(5.28)} \]

where

\[ A = \frac{1}{2} \rho SC_{D_0} \]  
\[ \text{(5.29)} \]

\[ B = \frac{2KW^2}{\rho S} \]  
\[ \text{(5.30)} \]

\[ C = \left( \frac{\rho}{\rho_o} \right)^{0.9} \]  
\[ \text{(Turbojet; Troposphere)} \]  
\[ \text{(5.31)} \]

\[ C = \left( \frac{\rho_{11000}}{\rho_o} \right)^{0.9} \left( \frac{\rho}{\rho_{11000}} \right) \]  
\[ \text{(Turbojet; Stratosphere)} \]  
\[ \text{(5.32)} \]
In these equations, the parameter are as follows: $S$, wing area, $W$, aircraft weight, $C_{D0}$, aircraft zero lift drag coefficient, $K$, induced drag correction factor, $\rho$, air density at any altitude, $\rho_o$, air density at sea level, and $\rho_{11000}$, air density at 11,000 meters altitude. The only unknown in equation 5.28 is the maximum speed. This equation is an algebraic nonlinear equation with the order of 4. When solving this equation, we will have four solutions. Only one solution is acceptable that is frequently the highest.

\[
C = \left( \frac{\rho}{\rho_o} \right)^{1.2}
\]

(Turbofan engine) (5.33)

Figure 5.12. The generic variations of maximum speed (true and equivalent) versus altitude

Figure 5.12 demonstrates the typical variations of the maximum speed versus flight altitude for a jet aircraft. In this figure, variations of true airspeed ($V_T$) and equivalent airspeed ($V_E$) are shown separately. As indicated earlier, the equivalent airspeed is always decreasing with altitude, but the true airspeed increases at the beginning, but decreases afterward. Therefore the maximum true airspeed has an absolute maximum (i.e., the maximum of the maximum airspeed). Based on this graph, every aircraft has a maximum speed at each altitude; however, the maximum of maximum speed occurs only at one altitude. This altitude depends on such parameters as aircraft weight, engine power and thrust, aircraft zero lift drag coefficient, and configuration.

One of the reasons why airplanes are flying at high altitude is the lower cost of flight. This is due to a lower fuel consumption with higher speed at high altitude; that also leads to a longer range. Table 5.3 demonstrates maximum speed of several jet aircraft. With the current technology, the highest speed of supersonic aircraft reaches Mach 3 and in special configuration could pass Mach 5. The maximum speed of a high subsonic aircraft is about Mach 0.95.

The supersonic transport airplane Concorde had a cruising airspeed of Mach of 2.2. It is one of only two SSTs to have entered commercial service; the other was the Tupolev Tu-144. The Concorde retired in 2,003 after a crash during take-off. On Tuesday, 25th July 2000 the very first fatal accident involving Concorde occurred with Concorde 203, out bound from Paris to New York.
It crashed 60 seconds after take-off after suffering tire blow out that caused a fuel tank to rupture. This started a sequence of events that caused a fire which eventually lead to 2 engines failing and the aircraft crashing. All 109 people (100 passengers and 9 crew members) on board were killed. Furthermore, 4 people in a local hotel on the ground were also killed. Figure 5.13 illustrates the reconnaissance aircraft Lockheed SR-71 Blackbird with a maximum speed of more than Mach 3.

<table>
<thead>
<tr>
<th>No</th>
<th>Aircraft Type</th>
<th>T (kN)</th>
<th>W_TO (N)</th>
<th>Altitude (ft)</th>
<th>V_max Knot</th>
<th>V_max Mach</th>
<th>Vc (knot)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Gulfstream G650 Business jet</td>
<td>2×71.6</td>
<td>450,000</td>
<td>41,000</td>
<td>516</td>
<td>0.9</td>
<td>488</td>
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<tr>
<td>2</td>
<td>Manchang Q-5M Fighter</td>
<td>36.8</td>
<td>120000</td>
<td>36000</td>
<td>688</td>
<td>1.2</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>Venga TG-10 Fighter</td>
<td>13</td>
<td>28,800</td>
<td>SL</td>
<td>485</td>
<td>0.72</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>Mirage F1 Attack</td>
<td>70.6</td>
<td>162,000</td>
<td>high</td>
<td>-</td>
<td>2.2</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>Rafale Fighter</td>
<td>2×71.2</td>
<td>140,000</td>
<td>-</td>
<td>800</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>Microjet 200B Trainer</td>
<td>2.6</td>
<td>13000</td>
<td>18000</td>
<td>250</td>
<td>-</td>
<td>210</td>
</tr>
<tr>
<td>7</td>
<td>Airbus 300 Transport</td>
<td>2× 262</td>
<td>1650,000</td>
<td>30000</td>
<td>-</td>
<td>-</td>
<td>484</td>
</tr>
<tr>
<td>8</td>
<td>Alphajet Trainer</td>
<td>2×14.1</td>
<td>80000</td>
<td>32800</td>
<td>-</td>
<td>0.86</td>
<td>-</td>
</tr>
<tr>
<td>9</td>
<td>Northrop F-5A Fighter</td>
<td>2×18.5</td>
<td>93790</td>
<td>36000</td>
<td>-</td>
<td>1.4</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>Tupolev Tu-28P Attack</td>
<td>120</td>
<td>450000</td>
<td>36000</td>
<td>1000</td>
<td>1.75</td>
<td>-</td>
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<tr>
<td>11</td>
<td>Concorde Transport</td>
<td>4×142</td>
<td>1,850,000</td>
<td>60000</td>
<td>-</td>
<td>2.3</td>
<td>M=2.04</td>
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<td>12</td>
<td>Tupolev Tu-144 Transport</td>
<td>4×172</td>
<td>1800000</td>
<td>52000</td>
<td>-</td>
<td>-</td>
<td>M=2.35</td>
</tr>
<tr>
<td>13</td>
<td>Fokker F-28 Transport</td>
<td>2×43.8</td>
<td>321,150</td>
<td>23000</td>
<td>-</td>
<td>-</td>
<td>445</td>
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<td>14</td>
<td>Rockwell B-1 Lancer Bomber</td>
<td>4×133</td>
<td>1768100</td>
<td>500</td>
<td>650</td>
<td>-</td>
<td>562</td>
</tr>
<tr>
<td>15</td>
<td>Mikoyan Mig-25 Fighter</td>
<td>2×108</td>
<td>350000</td>
<td>-</td>
<td>-</td>
<td>2.8</td>
<td>-</td>
</tr>
<tr>
<td>16</td>
<td>Boeing E-3 Sentry Early warning</td>
<td>4×93.4</td>
<td>1474170</td>
<td>-</td>
<td>460</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>17</td>
<td>Learjet 55 Transport</td>
<td>2×16.5</td>
<td>88450</td>
<td>3000</td>
<td>477</td>
<td>-</td>
<td>455</td>
</tr>
<tr>
<td>18</td>
<td>Grumman A-6 Attack</td>
<td>2×41.4</td>
<td>273970</td>
<td>SL</td>
<td>560</td>
<td>-</td>
<td>412</td>
</tr>
<tr>
<td>19</td>
<td>Aermacchi MB-326 Light Trainer</td>
<td>15.17</td>
<td>45770</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>430</td>
</tr>
<tr>
<td>20</td>
<td>Lockheed Martin F-16 Fighting Falcon Fighter</td>
<td>131.6</td>
<td>123,310</td>
<td>40,000</td>
<td>-</td>
<td>2.1</td>
<td>-</td>
</tr>
<tr>
<td>21</td>
<td>Boeing 777 Transport</td>
<td>2×342</td>
<td>2,993,700</td>
<td>3,600</td>
<td>-</td>
<td>-</td>
<td>M=0.87</td>
</tr>
<tr>
<td>22</td>
<td>Cessna 525 GA</td>
<td>2×8.45</td>
<td>47,170</td>
<td>33,000</td>
<td>-</td>
<td>0.7</td>
<td>383</td>
</tr>
<tr>
<td>23</td>
<td>Lockheed Martin F-22 Raptor Stealth Fighter</td>
<td>2×155</td>
<td>272,160</td>
<td>30,000</td>
<td>-</td>
<td>2.25</td>
<td>1.82</td>
</tr>
<tr>
<td>24</td>
<td>Northrop Grumman RQ-4 Global Hawk UAV</td>
<td>34</td>
<td>143,450</td>
<td>60000</td>
<td>-</td>
<td>-</td>
<td>310</td>
</tr>
<tr>
<td>25</td>
<td>Eurofighter Typhoon Fighter</td>
<td>2×60</td>
<td>230,456</td>
<td>55,000</td>
<td>-</td>
<td>M=2</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5.3. Cruise speed, and maximum speed of several jet aircraft

The ratio between the aircraft weight and the wing area (W/S) is called wing loading. The wing loading is an important parameter in aircraft performance evaluation. For example for an
Aircraft with a weight of 2650 lb, and wing area of 160 ft\(^2\), the wing loading is 2650/160 or 17.67 pond per square foot (lb/ft\(^2\)). Another term that appears in many performance equations is “T/W” which is called thrust-to-weight ratio. This ratio for majority of jet aircraft is less than one (about 0.2 to 0.3). For a vertical take-off and landing (VTOL) aircraft, this ratio is more than unity.

Two terms of “W/S” and “T/W” are fundamental parameters in aircraft performance, and influence many performance parameters such as the maximum speed (see equation 5.27). As the wing loading and thrust-to-weight ratio are increased, the maximum speed will be improved.

There are two limits for the maximum velocity of an aircraft; one lower limit, and one upper limit. The maximum speed of an aircraft should be always higher than the stall speed, since no non-VTOL; fixed-wing; heavier-than-air aircraft may have a sustained level flight with a speed lower than the stall speed.

Thus, if you experienced such result in your calculations, it implies that the engine is not powerful enough.

\[ V_{\text{max}} \geq V_s \]  
(5.34)

There is no lower limit for the maximum level velocity of a VTOL aircraft and a rotary wing aircraft.

![Lockheed SR-71 Blackbird](image)

**Figure 5.13.** Reconnaissance aircraft Lockheed SR-71 Blackbird

The upper limit of the maximum velocity of an aircraft is often a structural limit. This limit which also referred to as the never-exceeded velocity \(V_{\text{NE}}\) restricts an aircraft to fly beyond the point at which the structure will be damaged. At a speed beyond \(V_{\text{NE}}\), the bending moment on the structure (e.g., on the wing root), will be higher the design limit of the aircraft structure.
Moreover, the temperature at the critical locations of the structure (e.g., wing leading edge, or fuselage nose) may exceed the thermal limit.

**Example 5.4**  
A large jet transport aircraft with a mass of 165,000 kg, an engine thrust of 320 kN, a wing area of 260 m$^2$ has a zero lift drag coefficient of 0.02, and $K = 0.05$. Determine the maximum speed at sea level.

**Solution:**  
Using equation 5.23 through 5.26, we have:

\[
A = \frac{1}{2} \rho SC_D = 0.5 \times 1.225 \times 260 \times 0.02 = 3.185 
\]

\[
B = \frac{2 K W^2}{\rho S} = \frac{2 \times 0.05 \times (165000 \times 9.81)^2}{1.225 \times 260} = 822614701 
\]

\[
C = \left( \frac{\rho}{\rho_o} \right)^{0.9} = 1 \quad \text{(sea level)} 
\]

Now we substitute these coefficients into equation 5.23:

\[
AV_{\text{max}}^2 + \frac{B}{V_{\text{max}}^2} - CT_{\text{max}} = 0 \Rightarrow 3.185 V_{\text{max}}^2 + \frac{822614701}{V_{\text{max}}^2} - 320000 = 0 
\]

This algebraic equation has four solutions; the only acceptable solution for this equation is 285.8. Therefore:

\[
V_{\text{max}} = 285.8 \text{ m/sec} = 556 \text{ knot} = 0.84 \text{ Mach} 
\]

**5.3.2. Minimum-Drag Speed**  
Another interesting velocity in a straight level flight is the speed at which the aircraft generates the minimum drag. Since in a level flight, the thrust is equal to the aircraft drag, the minimum drag implies the minimum thrust, and subsequently, a lower fuel consumption.

The equation 2.4 implies that drag force is a direct function of the airspeed. A quick look at this equation initially reveals that as the velocity increases, the drag increases. This conclusion is not correct for all speeds.

The reason is as follows. When a pilot is going to increase the speed, the thrust must be increased. If the altitude is held constant, the aircraft angle of attack must be reduced. The reduction in the angle of attack means the reduction in the lift coefficient, and consequently, a reduction in induced drag coefficient ($C_{Di}$). Recall that, the drag force has two parts: induced drag; and zero lift drag.

Therefore, as the speed is increased, the induced drag coefficient is decreased. However, the increase in velocity leads to a more induced drag ($D_i$), while reduction in induced drag coefficient, results in a reduction in induced drag ($D_i$). The overall result is usually a reduction in the induced drag. In addition, the increase in the airspeed, results in an increase in the zero lift drag ($D_o$). Thus, an increase in the airspeed increases the zero lift drag ($D_o$), while decreases the induced drag ($D_i$). As the figure 5.5 demonstrates, the drag force has a minimum value when the velocity is varied. This speed is called the minimum drag speed ($V_{\text{minD}}$). It is evident
that the flight of an aircraft at this speed needs a minimum thrust for a level flight. Consequently, this flight consumes the minimum fuel and has the lowest cost of flight.

It is concluded that when the objective of a flight is to minimize the fuel consumption, the pilot must select and fly with this speed. An example of such a mission would be aerial surveillance to monitor a border or a region. In this mission, the objective is not to rush to the destination, but is to be in the air as long as possible. Another example is a mission to maximize the endurance. This velocity is unique for any specific aircraft and is a function of flight altitude.

To calculate the minimum drag speed, it suffices to differentiate the equation 2.4 with respect to velocity and set it to zero. To do this derivation, we start with the equation of drag as follows:

\[ D = \frac{1}{2} \rho V^2 SC_D \]  

(2.4)

Since

\[ C_D = C_{D_o} + KC_L^2 \]  

(3.10)

and

\[ C_L = \frac{2mg}{\rho V^2 S} \]  

(2.6)

Substitution of equation 3.10 and 2.3 in the equation 2.4, results in:

\[ D = \frac{1}{2} \rho V^2 SC_{D_o} + \frac{2K(mg)^2}{\rho V^3 S} \]  

(5.35)

By differentiating equation 5.35 with respect to \( V \); and setting it equal to zero, we obtain

\[ \frac{\partial D}{\partial V} = \rho VSC_{D_o} - \frac{4K(mg)^2}{\rho V^3 S} = 0 \]  

(5.36)

or

\[ V^4 = \frac{4K(mg)^2}{(\rho S)^2 C_{D_o}} \]  

(5.37)

And finally

\[ V_{\text{min},o} = \sqrt[4]{\frac{2\sqrt{K}(mg)}{(\rho S)^2 C_{D_o}}} = \sqrt[4]{\frac{2mg}{\rho S} \sqrt{\frac{C_{D_o}}{C_{D_o}}} \left( \frac{K}{C_{D_o}} \right)^{\frac{1}{2}}} \]  

(5.38)

Comparing this equation with equation 2.3, we can rewrite the equation 5.38 in the following format:

\[ V_{\text{min},o} = \sqrt[4]{\frac{2mg}{\rho S C_{D_o}^\alpha}} \]  

(5.39)

The minimum drag speed is inverse function of air density, hence; the minimum drag speed is increased with altitude. This is a general trend. The equation 5.39 further implies that the minimum drag speed is decreased when the aircraft weight is decreased.

Comparison of equations 5.38 and 5.39, we can conclude that the lift coefficient at minimum drag speed is equal to square root of ratio of the zero lift drag coefficient \( (C_{D_o}) \) and induced drag factor \( (K) \). Hence
The equation 5.40 is a mathematical expression; the theoretical value of $C_{\text{Lmin}}$ must be within a practical flight limit. The value of $C_{\text{Lmin}}$ from equation 5.40 cannot be more than the aircraft maximum lift coefficient ($C_{\text{Lmax}}$). If the output of the equation is more than $C_{\text{Lmax}}$, ignore the outcome, and select a new value slightly less than $C_{\text{Lmax}}$.

It is interesting to find out what the relationship between lift and drag coefficients is when an aircraft is flying with the minimum drag speed. In order to determine this relationship, we divide lift coefficient to drag coefficient, and insert their equivalent terms, which is:

$$\left(\frac{C_L}{C_D}\right)_{\text{min}} = \sqrt{\frac{C_{Do}}{K}} = \frac{C_{Dc}}{C_{Dc} + K C_{Dc} \frac{1}{2C_{Dc}}}$$

which is simplified to:

$$\left(\frac{C_L}{C_D}\right)_{\text{min}} = \frac{1}{2 \sqrt{K C_{Dc}}}$$

Comparing equations 5.40 and 3.10, yields the following useful result:

$$C_{D_{\text{min}}} = 2C_{Dc}$$

This equation implies that the drag coefficient of an aircraft when flying with a velocity that corresponds to the minimum drag, is equal to twice the value for $C_{Dc}$.

There is an important flight safety point about minimum drag speed, which is revealed by a careful look at the figure 5.5. This figure is reproduced as figure 5.1 with a minor change. The speed stability is a crucial requirement for a safe flight, which states that; when a pilot reduces the engine throttle, the aircraft speed must be decreased; and when a pilot increases the engine throttle the aircraft speed must be increased. Moreover; if the pilot does not change the throttle, but the aircraft speed is changed because of a wind gust (either increased or decreased), the speed must return to its original value. As such, this aircraft is referred to have the "speed stability". Figure 5.14 implies that an aircraft has speed stability only at a speed beyond the minimum drag speed, and does not own speed stability when the airspeed is lower than the minimum drag speed. This fact reveals a danger zone in level a flight.

The resonating is as follows. When a wind gust reduces the aircraft speed due to a disturbance, the aircraft speed must be increased to return to the original speed. However, when an aircraft flies with a speed less than the minimum drag speed, a reduction in the airspeed leads in the reduction in the drag. Similarly; an increase in the airspeed will lead in an increase in the drag. Since the engine thrust is held constant, the difference between thrust and drag force results in the change in the speed.

When the airspeed is higher than the minimum drag speed, and a disturbance hits; the change is the airspeed is negative. This means that the aircraft returns to its initial trim speed, and the aircraft is safe (speed stability). However, when the airspeed is lower than the minimum drag speed, and a disturbance hits; the change is the airspeed is positive. This means that the aircraft does not return to its initial trim speed, and the aircraft is not safe (no speed stability).
Consider the case of a jet aircraft when the speed is lower than the minimum drag speed (say $V_1 < V_{\text{minD}}$). Now assume that a wind gust reduces the aircraft speed to $V_2$ which is lower than the original speed of $V_1$ (i.e., $V_2 < V_1$). Since the speed is decreased, the drag is decreased (equation 3.1). Consequently, since the drag is decreased, the speed is decreased further (left-hand side of the graph in figure 5.14). The story is continued until the speed reaches the stall speed, and even the reduction is continued. We know that an aircraft is not able to maintain level flight, if the speed is lower than stall speed. Thus, this incident causes the aircraft to stall first and then have a fatal accident.

This is an unstable condition. The pilot must be careful when flying at a speed lower than the minimum drag speed. As soon as the pilot feels that the gust has reduced the speed, (s)he must increase the throttle in order to maintain the original speed. The good news is that for most aircraft the minimum drag speed is lower than the stall speed. So, it is impractical to fly with minimum drag speed for majority of aircraft.

For those aircraft that their minimum drag speed is higher than the stall speed, pilot has a very crucial role during take-off and landing operations. This is one of the reasons of higher crash rate at take-off and landing compared with other flight conditions. For those aircraft that their minimum drag speeds are lower than the stall speed, a safe minimum drag speed is selected (considered) to be about 10 to 20 percent higher than stall speed.

$$V_{\text{minD}} = k V_s$$ (5.45)

where

$$1.1 < k < 1.2$$ (5.46)
Example 5.5
A small jet aircraft has the following features:
\[ m = 2,500 \text{ kg}, \quad S = 20 \text{ m}^2, \quad C_{D_0} = 0.03, \quad K = 0.06, \quad V_s = 72 \text{ knot} \]
a. At what speed this aircraft must cruise; such that the engine produces the minimum thrust at 10,000 ft?
b. Repeat part a, when aircraft is flying at sea level.

Solution:
The speed at which the engine is producing a minimum thrust is the minimum drag speed. This speed is determined using equation 5.34. The lift coefficient for a minimum drag speed is calculated through equation 5.40 as follows:

\[
C_{L_{\text{min}}D} = \sqrt{\frac{C_{D_0}}{K}} = \sqrt{\frac{0.03}{0.06}} = 0.707 \quad (5.40)
\]
a. From atmospheric table in appendix B, the air density at the altitude of 10,000 ft is:
\[
\rho = 0.412 \frac{\text{kg}}{\text{m}^3}
\]
So,
\[
V_{\text{min},D} = \sqrt{\frac{2mg}{\rho SC_{L_{\text{min}}D}}} = \sqrt{\frac{2 \times 2500 \times 9.81}{0.412 \times 20 \times 0.707}} \Rightarrow V_{\text{min},D} = 91.7 \frac{m}{\text{sec}} = 178.3 \text{ knot} \quad (5.39)
\]
b. At sea level:
\[
V_{\text{min},D} = \sqrt{\frac{2mg}{\rho SC_{L_{\text{min}}D}}} = \sqrt{\frac{2 \times 2500 \times 9.81}{1.225 \times 20 \times 0.707}} \Rightarrow V_{\text{min},D} = 53 \frac{m}{\text{sec}} = 103.4 \text{ knot} \quad (5.38)
\]
It is observed that the minimum drag speed is increased with altitude. For both altitudes, the minimum drag speed is greater than the stall speed, which is OK.

It is interesting to calculate the minimum thrust an aircraft requires to have a steady level flight. The answer is obtained by inserting the minimum drag velocity (equation 5.39), the drag coefficient corresponding to the minimum drag velocity (equation 5.40), and the lift coefficient corresponding to the minimum drag velocity (equation 5.40), into the drag equation (equation 3.1; \( D = 1/2 \rho V^2 SC_D \)):

\[
T_{\text{min}} = D_{\text{min}} = \frac{1}{2} \rho V^2_{\text{min},D} S C_{L_{\text{min},D}} = \frac{1}{2} \rho \left( \sqrt{\frac{2mg}{\rho SC_{L_{\text{min}}D}}} \right)^2 S(2C_{L_{\text{D}}}) = \frac{1}{2} \rho \left( \frac{2mg}{\rho S \sqrt{K}} \right)^2 S(2C_{L_{\text{D}}})
\]

This simplifies to:
\[
T_{\text{min}} = 2W \sqrt{K C_{P_{\text{D}}}} \quad (5.46a)
\]
However, as in equation 5.24, the term \( \sqrt{K C_{P_{\text{D}}}} \) is equal to \( 1/2(L/D)_{\text{max}} \), thus, the minimum thrust will be:
\[
T_{\text{min}} = \frac{W}{(L/D)_{\text{max}}} \quad (5.46c)
\]
This is another proof for equation 5.11.
5.3.3. Maximum Lift-to-Drag Ratio Speed

The best aerodynamic efficiency of an aircraft is the when it cruise with a velocity such that delivers the highest value of lift-to-drag ratio \( \left( V_{(L/D)_{\text{max}}} \right) \). This velocity is of interest to a pilot whom the flight cost is of primary concern. To derive an expression for such velocity, we compare two equations of 5.19 and 5.42. The right hand side of equation 5.42 is exactly the same as right hand side of the equation 5.19. Therefore, the lift-to-drag ratio at the minimum drag speed is exactly the same as the maximum lift-to-drag ratio:

\[
\left( \frac{C_L}{C_D} \right)_{\text{min}} = \left( \frac{C_L}{C_D} \right)_{\text{max}}
\]

This implies that the minimum-drag speed is the speed at which the lift-to-drag ratio has the maximum value. This relationship is sketched in Figure 5.10. Thus, the velocity for minimum drag (i.e., minimum required thrust) is also the velocity for maximum L/D.

\[
V_{(L/D)_{\text{min}}} = V_{(L/D)_{\text{max}}}
\]

Thus, from equation 5.38, we can write:

\[
V_{(L/D)_{\text{max}}} = \sqrt{\frac{2mg}{\rho S \sqrt{\frac{C_{D_{\text{o}}} + C_{D_{\text{ind}}} K}}}}
\]

Comparing this equation with equation 2.3, we can rewrite the equation 5.49 in the following format:

\[
V_{(L/D)_{\text{max}}} = \sqrt{\frac{2mg}{\rho S C_{L_{(L/D)_{\text{max}}} \text{max}}}}
\]

The maximum L/D speed is inverse function of air density, hence; the maximum L/D speed is increased with altitude. This is a general trend. The equation 5.50 further implies that the maximum L/D speed is decreased when the aircraft weight is decreased.

Comparison of equations 5.49 and 5.50, we can conclude that the lift coefficient at maximum L/D speed is equal to square root of ratio of the zero lift drag coefficient \( C_{D_{\text{o}}} \) and induced drag factor \( K \). Hence

\[
C_{L_{(L/D)_{\text{max}}}} = \sqrt{\frac{C_{D_{\text{o}}}}{K}}
\]

The definitions and derivations of other specific speeds (i.e., maximum-range speed, maximum-endurance speed, and the speed for maximum ceiling) are presented in the following three sections.

5.4. Range

The range is considered as one of the most important parameters in civil aircraft performance and design. This is a first priority for transport aircraft, but the second priority for fighters. By definition, range is the total distance that an aircraft can fly with full fuel tank and without refueling. This consists of take-off, climb, cruise, descend, and landing (figure 5.15) and does not include the wind effect (either positive or negative). Usually, the cruise segment is the longest. The range is measured with respect to the ground. In a civil airplane, the definition is the maximum ground distance with full fuel tank minus the reserve fuel.
The reserve fuel is considered when the situation for destination airport is not safe for landing, thus the aircraft has to fly to another close airport with the remaining fuel. But for a military airplane, range is defined as two-way flight distance that includes take-off, climb, cruise, descend, maneuver, mission accomplishment (e.g., fight, bomb, and reconnaissance), climb again, return cruise, descend and landing (figure 5.1). It is also referred to as radius of operation or radius of action.

The significance of this performance criterion is appreciated when we consider the distance between cities and capitals of different countries. Each aircraft has limited range capabilities with specific speed and specific flight altitude. The flight with different flight conditions (e.g., altitude, speed) results in a different range. Therefore when we are talking about range, it automatically means the maximum range that is with the best flight condition to provide the maximum range. The range of an aircraft at different altitudes is not the same. Similarly, the range of an aircraft with different speed is not the same either. However, each aircraft has a unique maximum range, with the maximum take-off weight. This maximum range happens when flies at a specific (optimum) altitude and a specific (optimum) airspeed.

When Charles Lindbergh made his spectacular solo flight across the Atlantic Ocean on May 20-21, 1927, he could not have cared less about maximum velocity, maximum rate of climb, or maximum endurance. Uppermost in his mind was the maximum distance (i.e., range) he could fly on the fuel carried by the single-piston-engine aircraft “Spirit of St. Louis”. The range of 5834 km (distance from New York to Paris) was the main concern to Mr. Lindbergh in thus amazing flight (beside navigation and sleep).

In this section, we will first review various range definitions; and then derive relevant relationships, and finally the application of each equation is discussed. Table 5.4 demonstrates range of several jet aircraft. It is noticeable that the difference in range is directly related to their difference in their fuel capacity. Aerial refueling is a technique to increase the rage of a fighter. In this case, a tanker aircraft is flying in the region and when the fighter needs fuel, it must use its probe to connect to tanker’s drogue. During cold war era, several military aircraft such as Lockheed SR-71 Blackbird (Figure 5.13), Lockheed F-117 Nighthawk had aerial refueling to be able to accomplish their mission in a longer range.

Another technique to increase range is to use the external tank(s). These external tanks are often dropped and left behind when their fuel were consumed, since they are not permanent tanks. The record of longest range belongs to a two piston propeller engine aircraft Rutan Model 76 Voyager; and a single turbofan engine aircraft Scaled Composites Model 311 Virgin Atlantic GlobalFlyer [9]. Its range is equal to the circumference of the Earth (about 38,000 km), since it could circle the globe. This incident will be explained in detail in the next chapter.

Another long rage jet aircraft is Northrop Grumman RQ-4 Global Hawk [8]; an unmanned aerial vehicle (UAV) with a mass of 11,600 kg, a wing area of 50.2 m² is equipped with a turbofan engine with a maximum thrust of 31.4 kN. This surveillance aircraft has a range of over 14,000 km with a cruise speed of 310 knot and service ceiling of 60,000 ft.

5.4.1. Definition

There are several types of range with different definitions in the literature. Here four important types are introduced.
Safe Range: Safe Range is the maximum distance between two airfields that an aircraft can fly regularly without any problem. The safe range consists of take-off, climb, cruise, descend, and landing. In this type, the effect of wind (headwind or tailwind) is not considered; and it is assumed that the tank is full of fuel at the beginning. For safety reason, there must be a reserve fuel remained in the tank at the end (i.e., after landing). Minimum reserve fuel requirements are established by the Federal Aviation Regulations. The reserve fuel is often equivalent to either 20% of the total fuel; or 45 minutes of flight. This is a real case, but since the calculation of safe range is not an easy task, other definitions are offered.

Still Air Range (SAR): In the calculation of SAR (or net SAR), it is assumed that the flight begins with take-off; flight is continued until all fuel is consumed; and there is no landing. It is also imagined that the air is still, i.e., there is no wind during the flight. This definition is not a real flight, but instead, the calculation of SAR is more convenient. Figure 5.16 demonstrates this definition. The next type offers a much simpler calculation.

Gross Still Air Range (GSAR): The Gross Still Air Range does not include any segment other than cruising flight; and it is also assumed that the flight begins with a full fuel tank. By definition, this flight ends in the air until the entire fuel is consumed. This is the definition that we are interesting in; and the calculation of GSAR is straight forward. Like, SAR, the GSAR ignores the influence of wind to the range. For technical reason, we know that the weight of aircraft is constantly decreasing, since the fuel is consumed by the engine. For ease of mathematical derivation, we resort to the fourth definition that is specific range. GSAR is sometime is referred to as “cruise range”.

Figure 5.15. Range of a civil and military aircraft
- **Specific Range (SR):** Specific Range is defined as the distance flown divided by amount of fuel that is consumed. In a simple word, the value of the miles per pound of fuel is called specific range. This is analogous to the mileage of an automobile, especially when it is expressed in miles per gallon (MPG).

![Diagram of aircraft performance analysis](image)

**Figure 5.16. Net still air range and gross still air range**

To derive an expression for range, we employ the specific range. In terms of math language, specific range is the differentiation of the flown distance (X) with respect to the aircraft weight (W); or in effect, fuel weight:

\[
SR = \frac{dX}{dW}
\]

(5.52)

The units of specific range are: \( \frac{km}{N} \); \( \frac{nm}{lb} \); \( \frac{mile}{lb} \). Among four definitions, the first one is the most accurate, but hard to calculate, but the last one is the most unrealistic, but easiest to derive. The third definition (GSAR) is relatively easy to handle, thus we will derive several relationships based on GSAR. At the end of this section (5.4), a discussion of how to include the influence of wind on the range is presented.

### 5.4.2. Calculation of Range

The range is of distance type, and the distance is defined as the velocity times the duration of motion. We begin the derivation of specific range with the definition of velocity. The instantaneous velocity is defined as differentiation of the distance traveled with respect to time (t).

\[
V = \frac{dX}{dt}
\]

(5.53)

On the other hand, fuel mass flow rate (Q_f) is defined as the differentiation of the aircraft weight (W) with respect to time:

\[
Q_f = \frac{dW}{dt}
\]

(5.54)

In chapter 4, we defined Specific Fuel Consumption (SFC) as the weight of fuel consumed per unit time, per unit thrust, so:

\[
SFC = C = -\frac{dW/dT}{T} = -\frac{Q_f}{T}
\]

(5.55)

The minus sign is added, because the rate of change of aircraft weight is a negative value and C is always treated as a positive quantity. Combining equations 5.48, 5.49, and 5.50, and substituting them into equation 5.52 yields:
\[ SR = \frac{dX}{dW} = \frac{V dt}{Q dt} = \frac{V}{Q} = -\frac{V}{CT} \]  
\[ (5.56) \]

In a cruising flight (with constant speed), the drag force (D) must equal to the engine thrust (T), therefore; T is replaced with D:

\[ SR = -\frac{V}{CD} \]  
\[ (5.57) \]

Now we multiply both the numerator and the denominator with lift force (L)

\[ SR = -\frac{VL}{CDL} \]  
\[ (5.58) \]

Recall that in a cruising flight, the lift force (L) is also equal to the weight of aircraft (W), so the L in denominator is replaced with W:

\[ SR = -\frac{VL}{CDW} = -\frac{V L/D}{CW} \]  
\[ (5.59) \]

This relationship is very informative. It implies that specific range is a function of four parameters: velocity, specific fuel consumption, aircraft weight (in effect, fuel weight), and lift-to-drag ratio. Based on the definition of specific fuel consumption, it is correct to assume that specific fuel consumption is constant for a jet engine. So, specific range is a function of velocity multiplied by lift-to-drag ratio. This implies that in order to increase specific range, both velocity and lift-to-drag ratio must be increased simultaneously. But we know that the lift-to-drag ratio is a function of velocity and its maximum only happens at only one particular velocity. Figure 5.17 illustrates a GA aircraft Cessna Citation IISP with turbofan engines.

**Figure 5.17. Cessna Citation IISP**

In practice, in order for an aircraft performance engineer to find the maximum specific range, (s)he first determines the lift coefficient (CL) corresponding to the maximum lift-to-drag ratio ((L/D)max). In practice, a slightly lower CL is chosen. The next step would be to find the highest altitude corresponding to this CL. Figure 5.18 demonstrates the generic variations of specific range versus velocity for several altitudes for a jet aircraft. Note that each curve has a unique maximum. As soon as the velocity (in terms of Mach number) goes beyond the value such that the percentage of increase in drag force (due to wave drag) is more than the percentage in increase in velocity, specific range begins to decrease. At this altitude and at this velocity, the specific range is at its optimum value. It is often recommended to choose the cruising speed to be close to this value.
For instance; consider a transport aircraft with a mass of 100,000 kg, a wing area of 240 m², a $C_{D_0}$ of 0.02 at low subsonic Mach number; a $C_{D_0}$ of 0.04 at high subsonic Mach number equipped with two turbofan engines. The variations of drag of this aircraft versus velocity is similar to what is shown in figure 5.5. To determine the maximum specific range, we examine three flight altitudes of 8,000 m, 9,000 m, and 10,000 m. We observe that the specific range at 8,000 m is lower that at 9,000 ft (0.114 is less than 0.122); and the specific range at 9,000 m is lower that at 10,000 ft (0.122 is less than 0.113). Then, it is concluded that maximum specific range happens at 10,000 m with the corresponding speed of Mach 0.75. It is evident that this is true only if the initial aircraft mass is 100,000 kg. In another word, when the take-off mass is different than this value, the flight condition for maximum specific range would be different.

Flying exactly at a speed less than the minimum drag speed causes the speed instability. When an aircraft is flying at $V_{(L/D)\text{max}}$, a speed decrease due to a disturbance raises the drag and tends to slow the airplane further. Continual pilot throttle corrections are needed to maintain the cruise speed. Such an airplane lacks speed stability. If the airplane is cruising faster than $V_{(L/D)\text{max}}$, an inadvertent slowing will decrease the drag. Then, thrust exceeds drag and tends to accelerate the aircraft automatically, returning to the planned cruise speed.
The expression $M(L/D)$ is a measure of the specific range capability due to the aerodynamic characteristics of a jet aircraft. Figure 5.19 illustrates curves of $M(L/D)$ versus cruise Mach numbers ($M$) for several altitudes for the aircraft introduced in Figure 5.18. Since $V = Ma$, equation 5.43, the specific range equation for jets, can be reformatted as

$$SR = -a \frac{M}{C} \frac{L}{D} \frac{1}{W}$$  \hspace{1cm} (5.60)

The curves of $M(L/D)$ versus cruise Mach number graphically show the previously discussed fact that specific range is increased by high speed. The maximum $M(L/D)$ is obtained at a Mach number of 0.77 and an altitude of 10,000 m. Furthermore, the decrease in $M(L/D)$ is only 6.6% (10.6 to 9.9) if the altitude is reduced to 9,000 m at $M = 0.77$. This figure also indicates that this jet aircraft need not be flown at an exact altitude in order to achieve the maximum range. Figures 5.18 and 5.19 illustrate that the speed for maximum range is increased as the flight altitude is increased. Figures 5.18 and 5.19 are produced by a matlab file based on the technique introduced in this chapter. This implies a faster flight with a higher range for a jet aircraft.

Now we return to the discussion about the mathematical calculation of range. In general, performance can be evaluated based on two approaches:

1. Instantaneous or point performance

The point performance of an aircraft is the performance at a specified point on the flight path or at a specified instant of time. Mission performance primarily lies in determining the overall flight performance. Examples are: how far can a particular aircraft fly with a given amount of fuel or conversely, how much fuel is required to fly a desired range. This can be done by integrating the point performance over the interval between specified initial and final points, usually the start and end of the cruising flight.

The specific range is one measure of point performance, but our interest is about overall range or mission performance. To integrate, we use the equation 5.52. Conditions at the start of cruise will be identified by the subscript 1 and at the end of cruise by the subscript 2. By using the definition of specific range ($SR$), we can write:

$$R = \int_{0}^{R} dX = \int SR \, dW$$  \hspace{1cm} (5.61)

Since specific fuel consumption ($C$) can be assumed constant for a jet engine, equation 5.59 can be written as:

$$R = -\frac{1}{C} \left[ \frac{W_1}{W_2} \right] \frac{V}{D} \frac{L}{W}$$  \hspace{1cm} (5.62)

where $W_1$ is the weight of aircraft at the beginning of the flight and $W_2$ is the weight of aircraft at the end of flight. The variable $W_1$ is usually assumed to be the maximum take-off weight ($W_{TO}$) and the variable $W_2$ as the difference between the maximum take-off weight and the fuel weight. Equation 5.62 is a general equation for range for a jet aircraft. At any weight and altitude, the speed is associated with an angle of attack, and in deed, with a lift coefficient ($C_L$) that is:

$$C_L = \frac{2W}{\rho S V^2}$$  \hspace{1cm} (5.62a)

Considering the equation 2.3 in mind, the equation 5.62 has independent parameters that are: weight ($W$), velocity ($V$), altitude or its corresponding air density ($\rho$), and angle of attack, or its associated lift coefficient ($C_L$). In order to solve the integration and come up with a closed-
form solution, we need to set a few simplifying assumptions. Since the fuel is consumed during flight, the aircraft weight is constantly decreased during the flight. In order to maintain a level flight, we have to decrease the lift as well. Of the many possible solutions only three are more practical and will be examined. In each case, two flight parameters will be held constant throughout cruise. The three options of interest for continuous decrease of the lift during cruise are (Figure 5.20):

1. Decreasing flight speed (Constant-altitude, constant-lift coefficient flight)
2. Increasing altitude (Constant-airspeed, constant-lift coefficient flight)
3. Decreasing angle of attack (Constant-altitude, constant-airspeed flight)

For each flight program, the integral equation (5.62) will be set up and then only the final range equation will be shown and discussed. In the first option, the velocity must be reduced with the same rate as the aircraft weight is decreased. In the second solution, the air density must be decreased; in another word, the flight altitude must be increased. The third option offers the reduction of aircraft angle of attack; i.e., the reduction of lift coefficient. In terms of pilot operation, the first option is applied through throttle; and the third option is implemented through stick/yoke/wheel. In the second option, no action is needed by the pilot; the aircraft will gradually gain height (climbs).

Based on the safety regulations and practical considerations, the second option is option of interest for majority of aircraft. The reason will be explained later. In general, when flight is conducted under the jurisdiction of Federal Aviation Regulations, the accepted flight program is the constant altitude-constant airspeed flight program.

5.4.2.1. Flight program 1. Constant-altitude, constant-lift coefficient flight
In this option, the velocity will be reduced via throttle as the aircraft weight is decreased. This flight program to be examined is the constant altitude-constant lift coefficient flight program. Since the lift coefficient is held constant throughout cruise, the lift-to-drag ratio (L/D) will also be constant. It is convenient, therefore, to express the instantaneous drag as the ratio of the instantaneous weight to the instantaneous lift-to-drag ratio. Since L/D and C are assumed constant throughout cruise, they are taken out of the integral in the Equation 5.62 as
\[ R = -\frac{1}{C D} \int_{W_i}^{W_f} \frac{dW}{V} \]  

Performing the integration using the equation for the velocity \( V \) yields the range equation as follows:

\[ R = -\frac{1}{C D} \int_{W_i}^{W_f} \frac{2W}{\rho S C_L} \frac{dW}{W} = -\frac{1}{C D} \int_{W_i}^{W_f} \frac{2}{\rho S C_L} dW \]  

The integration results in:

\[ R = \frac{2}{\rho S C_L} \left[ 2\left(\sqrt{W_i} - \sqrt{W_f}\right) \right] = \frac{2}{\rho S C_L} \left[ \frac{2W_i}{\sqrt{W_i}} \left( 1 - \frac{W_f}{W_i} \right) \right] \]  

which finally simplifies to:

\[ R = \frac{2}{C D} V_i \left( 1 - \sqrt{1 - \frac{W_f}{W_i}} \right) \]  

where \( V_i \) is the initial airspeed and \( W_i \) is the gross weight of aircraft at the start of cruise. The \( W_f \) denotes the fuel weight and is obtained by:

\[ W_f = W_i - W_2 \]  

The aircraft final weight will be:

\[ W_2 = W_i \left( 1 - \frac{W_f}{W_i} \right) \]  

It can be seen from equation 2.3 that the airspeed must be decreased as fuel is consumed if \( C_L \) is to be constant as the weight decreases along the flight path. To determine the final airspeed \( V_2 \); we know that the aircraft weight is equal to the lift at the beginning and end of flight:

\[ W_1 = L_1 = \frac{1}{2} \rho V_1^2 S C_L \]  

\[ W_2 = L_2 = \frac{1}{2} \rho V_2^2 S C_L \]  

Inserting equation 5.69 and 5.70 into equation 5.68 yields:

\[ \frac{1}{2} \rho V_2^2 S C_L = \frac{1}{2} \rho V_1^2 S C_L \left( 1 - \frac{W_f}{W_i} \right) \]  

which simplifies to:

\[ V_2 = V_i \left( 1 - \frac{W_f}{W_i} \right) \]  

Since lift coefficient \( (C_L) \) is held constant, equation 5.8 implies that the thrust must be constantly decreased (by constantly setting back the throttle) as the fuel is used (i.e., the gross weight decreases). The variations in the flight parameters for this flight program are shown in figure 5.21 as functions of the fuel weight fraction, which is a measure of the range flown.

The lift coefficient is constant throughout the flight; and based on equations 5.69 and 5.70; it is determined from:

\[ C_L = \frac{2W_i}{\rho V_1^2 S} = \frac{2W_2}{\rho V_2^2 S} \]  

(5.72b)
For the viewpoint of pilot control, there are three drawbacks to this flight program. The first is the need to continuously compute the airspeed along the flight path and to reduce the throttle setting accordingly. The second is that reducing the airspeed increases the flight times. The third is the fact that air traffic control rules require “constant” true airspeed for cruise flight, currently constant means ± 10 knots. The good news is that, current autopilot of large transport aircraft has solved part of this problem (i.e., no need for pilot calculation).

5.4.2.2. Flight program 2. Constant-airspeed, constant-lift coefficient flight

The second flight program to be examined is the constant airspeed-constant lift coefficient flight program, which is commonly referred to as cruise-climb flight. In this option, the air density will be automatically reduced as the aircraft weight is decreased. No pilot intervention is necessary. To evaluate cruise-climb flight, the level-flight equations is used with the assumption that the flight-path angle is sufficiently small. Thus the basic operating condition for cruise-climb flight is that the ratio of the instantaneous weight of the aircraft to the air density is constant.

Since both velocity (V) and lift-to-drag ratio (L/D) are constant, the variables C, V and L/D will be taken out of the integral, and the equation 5.62 can be written as

\[ R = -\frac{1}{C} V \frac{L}{D} \int \frac{dW}{W} \]  
(5.73)

The result of this integration is

\[ R = -\frac{V}{C D} \ln \left( \frac{W_2}{W_1} \right) \]  
(5.74)

or

\[ R = \frac{V}{C D} \ln \left( \frac{W_1}{W_2} \right) \]  
(5.75)

Since the weight of the aircraft at the end of flight is equal to initial weight minus fuel weight, so

\[ W_2 = W_i - W_f \]  
(5.76)

Therefore, equation 5.75 can be reformatted in terms of fuel weight as:

\[ R_2 = \frac{V(L/D)}{C} \ln \left( \frac{1}{1 - \left( \frac{W_f}{W_i} \right)} \right) \]  
(5.77)

This is the general form of what is known as the Breguet range equation. In order to keep both the airspeed and the lift coefficient constant as the weight of the aircraft decreases, equation 2.3 demonstrates that air density (\( \rho \)) must decrease in a manner so as to keep the ratio of the lift to weight constant. The only way that this can be done is to increase the altitude in an appropriate manner. Consequently, the aircraft will be in a continuous climb (thus, the name cruise-climb), which appears to violate the level-flight condition of a zero flight-path angle. It will be shown in a subsequent section that the cruise-climb flight-path angle is sufficiently small so as to justify the use of the level-flight equations and solutions for cruise-climb. The thrust required will decrease along the flight path in such a manner that the available thrust will decrease in an identical manner.
Therefore, cruise-climb flight requires no computations or efforts by the pilot. In an aircraft equipped with an autopilot, the aircraft cruise control will be implemented by the autopilot. After establishing the desired cruise airspeed, the pilot simply engages the Mach-hold mode (or constant-airspeed mode) on the autopilot; and the aircraft will slowly climb at the desired flight-path angle as the fuel is burned. The variations in the flight parameters along the flight path are shown in figure 5.21. Only under certain limited conditions; the cruise-climb flight is allowed by air traffic control. Concorde, the supersonic transport aircraft was allowed to employ this program, since it was flying at high altitude (above 50,000 ft).

The cruise altitude for a cruise-climb flight option is gradually increasing. The altitude at the end of cruise-climb flight (h₂) can be expressed in terms of both the initial altitude (h₁) and the fuel fraction. The derivation is provided in Section 5.7. The density ratio at the end of cruise-climb flight (σ₂) can be expressed in terms of both the initial density ratio and the fuel fraction: 

\[
\sigma_2 = \sigma_1 (1 - G) \quad (5.183)
\]

where, \(\sigma_1\) is air density ratio at the beginning of the cruise, and G is the fuel fraction. The climb angle is so small that can be ignored. When the air density at the end of cruise-climb flight (\(\sigma_2\)) obtained, one can utilize Appendix A or B to determine the final altitude.

The lift coefficient is constant throughout the flight; and based on the lift equation; it is determined from:

\[
C_L = \frac{2W_1}{\rho_1 V^2 S} = \frac{2W_2}{\rho_2 V^2 S} \quad (5.77b)
\]

**5.4.2.3. Flight program 3. Constant-altitude, constant-airspeed flight**

In this option, the angle-of-attack will be reduced via stick/wheel as the aircraft weight is decreased. The integral range equation 5.62 is repeated here for convenience:

\[
R = -\frac{1}{C \frac{W_1}{g_1}} \int V \frac{L}{D} \frac{dW}{W} \quad (5.62)
\]

Since the lift is equal to the weight during cruise, the L from numerator and W from denominator are eliminated. Thus the equation for this flight program can be simplified as:

\[
R = -\frac{V}{C \frac{W_1}{g_1}} \int \frac{dW}{D} \quad (5.78)
\]

Since the altitude and airspeed are constant in this flight option, the dynamic pressure (q) will be constant:

\[
q = \frac{1}{2} \rho V^2 = \text{const} \quad (5.79)
\]

Hence, the drag is written in this form:

\[
D = qSC_D = qSC_{D_0} + \frac{KW^2}{qS} \quad (5.80)
\]

Substituting equation 5.80 into equation 5.78, and moving the constant parameters (coefficients) out of the integration yields:

\[
R = \frac{V}{CqSC_{D_0}} \int \frac{-dW}{W_1 + \frac{KW^2}{q^2 S^2 C_{D_0}}} \quad (5.81)
\]

The result of integration would be:
\[ R = \frac{V}{CqSC_D} \sqrt{\frac{K}{q^2 S^2 C_D}} \left( \tan^{-1} \sqrt{\frac{K}{q^2 S^2 C_D}} W_1 - \tan^{-1} \sqrt{\frac{K}{q^2 S^2 C_D}} W_2 \right) \] (5.82)

Figure 5.21. Variation of flight parameters as a function of fuel weight ratio

The bracketed term represents the difference between two angles and can be written as

\[ \tan^{-1} \sqrt{\frac{K}{q^2 S^2 C_D}} W_1 - \tan^{-1} \sqrt{\frac{K}{q^2 S^2 C_D}} W_2 = \tan^{-1} \left[ \frac{\sqrt{KC_D} W_1 W_f}{\sqrt{KC_D} W_1 W_2} \right] \]

\[ \frac{1 - \frac{KC_D W_1 W_f}{D_1}} \] (5.83)
Since \( \frac{W_i}{D_i} = L_i / D_i = C_{t_i} / C_{d_i} \), the initial lift-to-drag ratio, and \( \sqrt{KC_{D_{0}}} = \frac{1}{\sqrt{2(C_{L}/C_{D})_{\text{max}}}} \) for a parabolic drag polar, we can rewrite equation 5.83 and substitute it in equation 5.82 to obtain the rage equation of this case as:

\[
R_3 = \frac{2V(C_L/C_D)_{\text{max}} \tan^{-1} \left[ \frac{(C_L/C_D)_i \frac{W_f}{W_i}}{2(C_L/C_D)_{\text{max}} \left(1 - KC_{L_i}(C_L/C_D)_i \frac{W_f}{W_i}\right)} \right]}{C}
\]

(5.84)

In this equation, the angle in the arc tangent term must be expressed in radian. In equation 5.84, \( C_{t_i} / C_{d_i} \) denotes the initial lift-to-drag ratio, and \( C_{t_i} \) denotes the initial lift coefficient:

\[
C_{t_i} = \frac{2W_i}{\rho V_i^2 S}
\]

(5.84b)

The final lift coefficient is readily obtained as:

\[
C_{t_2} = \frac{2W_2}{\rho V_2^2 S}
\]

(5.84c)

where \( W_2 \) is the final weight at the end of cruise. Throughout the flight (including at the beginning at the end of the flight), the lift equals weight:

\[
W_1 = L_1 = \frac{1}{2} \rho V^2 SC_{t_i}
\]

(5.84d)

\[
W_2 = L_2 = \frac{1}{2} \rho V^2 SC_{t_2}
\]

(5.84e)

where the relation between the aircraft weight at the beginning at the end of the flight is given by equation 5.68. Inserting equation 5.x and 5.x into equation 5.68 yields:

\[
\frac{1}{2} \rho V^2 SC_{t_2} = \frac{1}{2} \rho V^2 SC_{t_i} \left(1 - \frac{W_f}{W_i}\right)
\]

(5.84f)

which simplifies to:

\[
C_{t_2} = C_{t_i} \left(1 - \frac{W_f}{W_i}\right)
\]

(5.84g)

which yields the final lift coefficient as a function of the initial lift coefficient.

From the theoretical point of view, all three cases are realizable, but in practice; only the third case is acceptable, and approved for transport aircraft by FAA. This case is hard to follow, because the pilot must constantly decrease the angle of attack, through deflection of the elevator. However, for an aircraft equipped with an autopilot; this is an easy task. This case is the safest flight program among three possible programs. Figure 5.21 illustrates variations of several flight parameters as a function of fuel weight ratio (\( W_f/W_i \)) for these three cases.

The airliner Airbus A-320 (Figure 5.22) with a maximum take-off mass of 77,000 kg has an initial cruise altitude of 37,000 ft, while the maximum certified altitude is 39,800 ft. The business tri-jet aircraft Dassault Falcon 50 with a maximum take-off mass of 18,000 kg has an initial cruise altitude of 41,000 ft, while the maximum certified altitude is 49,800 ft.
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Table 5.4. Range of several jet aircraft

5.4.3. Speed for Maximum Range (V\text{max}_R)

In section 5.4.2 the derivation of Breguet range equation was introduced. This equation allows us to determine range of any jet airplane. There is a velocity parameter in this equation that must be known prior to range calculation. Maximum range velocity (V\text{max}_R) is the aircraft velocity that yields maximum range. There are basically three programs to reduce lift as the aircraft weight is decreased during a cruising flight.

1. Decreasing flight speed (Constant-altitude, constant-lift coefficient flight)
2. Increasing altitude (Constant-airspeed, constant-lift coefficient flight)
3. Decreasing angle of attack (Constant-altitude, constant-airspeed flight)

Since, only variable beside aircraft weight in the integration of equation 5.62 is the flight airspeed, these three programs may be classified into two groups: 1. Constant-speed cruising flight (programs 2 and 3); 2. Reducing-speed cruising flight (program 1). The mathematical calculations of the cruising speed for these two groups are different; so they are presented separately.

5.4.3.1. Constant-speed cruising flight

This section, covers the velocity to gain maximum range for the second and third flight programs (Constant-airspeed, constant-lift coefficient flight). In a cruising flight for maximizing range when the speed is kept constant, there are two options: 1. Reduce the lift coefficient, and 2. Increase the cruising altitude. For both of these programs, the mathematical solution is developed in the same...
way. To determine the speed, we simply differentiate Specific Range (SR) with respect to the velocity and set it equal to zero. Recall that the variation in the specific fuel consumption with the airspeed is small and can be assumed constant. As defined earlier, the Specific Range is mathematically a ratio of two variables (V and D) where the denominator (D) is also a function of another parameter (V).

\[ SR = \frac{V}{CD} \]  

(5.57)

For this differentiation, we need to employ the following differentiation technique:

\[ \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{(du/v) - (dv/u)}{v^2} \]  

(5.85)

Differentiation of equation 5.57 with respect to velocity (V) is as follows:

\[ \frac{d}{dV} (SR) = 0 \Rightarrow \frac{d}{dV} \left( -\frac{V}{CD} \right) = -\left[ \frac{CD}{(CD)^2} - \frac{CV}{(CD)^2} \left( \frac{dD}{dV} \right) \right] = 0 \]  

(5.86)

or

\[ \frac{d}{dV} \left( \frac{V}{CD} \right) = \frac{1}{CD} - \frac{V}{CD^2} \left( \frac{dD}{dV} \right) = 0 \]  

(5.87)

Thus, the condition for maximum range becomes:

\[ \frac{dD}{dV} - \frac{D}{V} = 0 \Rightarrow \frac{dD}{dV} = \frac{D}{V} \]  

(5.88)

where, we already had:

\[ D = \frac{1}{2} \rho V^2 S C_{D_v} + \frac{2K(mg)^2}{\rho V^2 S} \]  

(5.35)

\[ \frac{\partial D}{\partial V} = \rho V S C_{D_v} - \frac{4K(mg)^2}{\rho V^3 S} \]  

(5.36)

By substitution of equations 5.36 and 5.36 into this equation and after manipulation, we will obtain:

\[ \frac{2mg}{\rho V^2 S \sqrt{C_{D_v}}} = 1 \]  

(5.89)

![Figure 5.22. Airbus A320 with two turbofan engines](image-url)
This is the equation that yields maximum range speed for a jet aircraft. By comparing this equation with equation 5.38, we see that the maximum range velocity is always greater than the minimum-drag velocity. We note that \( V_{\text{maxR}} \) is inversely proportional to the square root of the density ratio and thus increases with altitude. In addition, \( V_{\text{maxR}} \) increases in direct proportion to the square root of the wing loading \((W/S)\). A doubling of the wing loading results in a 14 percent increase in range. Furthermore, \( V_{\text{maxR}} \) is inversely proportional to the \( \frac{1}{4} \) root of the zero-lift drag coefficient, and hence decreases with \( C_{D_0} \). Ref [13] illustrates that this velocity minimizes the number of pounds of fuel consumed per mile \((i.e., dW/dX)_{\text{min}} \). Due to this reference, this velocity is sometime called Carson’s speed.

In the case of the flight program 2; where the lift coefficient is held constant flight; the following can be concluded by comparing equation 5.90 with equation 5.39:

\[
C_{L,\text{max}} = \sqrt{\frac{C_{D_0}}{3K}} = 0.577 C_{l(\text{minimum})}\quad (5.91)
\]

and

\[
C_{D,\text{max}} = \frac{4}{3} C_{D_0}\quad (5.92)
\]

The lift-to-drag ratio for the maximum range would be

\[
\left( \frac{L}{D} \right)_{\text{max}} = \frac{C_{L,\text{max}}}{C_{D,\text{max}}} = \frac{\sqrt{C_{D_0}/3K}}{4/3C_{D_0}} = \frac{\sqrt{3}}{2 \sqrt{K C_{D_0}}} = 0.866 \left( \frac{L}{D} \right)_{\text{max}}\quad (5.93)
\]

In other word, lift-to-drag ratio when flying for maximum range is equal to 86.6 percent of the maximum lift-to-drag ratio.

5.4.3.2. Non Constant-speed Cruising Flight

This section, covers the velocity to gain maximum range for the first flight program. In this program, the flight speed is decreasing; and is a constant-altitude, constant-lift coefficient flight. In a cruising flight for maximizing range when the speed is reduced, there is no unique velocity to maximize the range. In both cases, we have to determine at least two velocities: 1. initial velocity, 2. final velocity.

For both cases, the initial speed is the special speed to deliver the maximum range and is determined as in equation 5.90:

\[
V_1 = V_{\text{maxR}} = \sqrt{\frac{2mg}{\rho S \sqrt{C_{D_0}/3K}}} = \sqrt{\left( \frac{2mg}{\rho S} \right) \left( \frac{3K}{C_{D_0}} \right)^{\frac{1}{2}}}\quad (5.94)
\]

The final velocity is found using the following equation; that we derived earlier in this chapter.

\[
V_2 = V_1 \sqrt{1 - \frac{W_f}{W_i}}\quad (5.72)
\]

For both cases, the speed throughout cruising flight may be determined accordingly.
5.4.4. Calculation of Maximum Range

This Section is devoted to the calculation of maximum range. There are basically three programs to reduce lift as the aircraft weight is decreased during a cruising flight: 1. Decreasing flight speed (Constant-altitude, constant-lift coefficient flight) 2. Increasing altitude (Constant-airspeed, constant-lift coefficient flight) 3. Decreasing angle of attack (Constant-altitude, constant-airspeed flight). These three flight program are treated separately.

5.4.4.1. Constant-altitude, constant-lift coefficient flight

In the case of constant altitude, constant lift coefficient (flight program 1), substitution of maximum range velocity into equation 5.66 results in the following maximum range:

\[
R_{\text{max}_1} = \frac{1.732V_{\text{max}_R} (L/D)_{\text{max}}}{C} \left[1 - \sqrt{1 - \frac{W_f}{W_t}}\right]
\]  

(5.95)

In this equation, \(V_{\text{max}_R}\) is set as the initial speed. Thus, the initial speed is determined from equation 5.94, and the final velocity from equation 5.72. In addition, the lift coefficient is determined from equation 5.91.

5.4.4.2. Constant-airspeed, constant-lift coefficient flight

The maximum range equation for the flight program 2; constant-airspeed, constant-lift coefficient flight is obtained by substitution of equations 5.90 and 5.93 into equation 5.72 as follows:

\[
R_{\text{max}} = R_{\text{max}_2} = \frac{0.866V_{\text{max}_R} (L/D)_{\text{max}}}{C} \ln \left[\frac{1}{1 - \frac{W_f}{W_t}}\right]
\]  

(5.96)

Note that this equation is based on the constant velocity, constant lift coefficient assumption (cruise-climb). In this flight program, the lift coefficient is determined from equation 5.91, and the speed is determined from equation 5.94.

5.4.4.3. Constant-altitude, constant-airspeed flight

In the case of constant-velocity, constant-altitude flight (flight program 3), the derivation of maximum range equation is derived from equation 5.84 as follows. According to equation 5.93, the ratio between “lift-to-drag ratio for maximum range” to “the maximum lift-to-drag ratio” is 0.866, so \(\frac{(C_L/C_D)_{\text{max}}}{2(C_L/C_D)_{\text{max}}} = \frac{0.866}{2} = 0.433\).

In addition, based on equation 5.91, the initial lift coefficient is:

\[
C_{\text{L_i}} = C_{L,\text{max}} = \sqrt{\frac{C_{D_i}}{3K}} = 0.577C_{L(\text{a})_{\text{max}}}
\]  

(5.96b)

Furthermore, based on equation 5.93, the initial lift-to-drag ratio is:

\[
(C_L/C_D)_i = \left(\frac{L}{D}\right)_{\text{max}} = 0.866(L/D)_{\text{max}}
\]  

(5.96c)

Using equations 5.24 and 5.23, and plugging into part of the equation 5.84; yields:

\[
KC_{\text{L_i}} (C_L/C_D)_i = K \left(0.577C_{L(\text{a})_{\text{max}}} \right) 0.866 \left(\frac{L}{D}\right)_{\text{max}} = K \left(0.577\right) \sqrt{\frac{C_{D_i}}{K}} \left(0.866\right) \frac{1}{2\sqrt{KC_{D_i}}}
\]

\[
= \frac{0.577 \times 0.866}{2} = 0.25
\]
Thus, the following equation yields the maximum range:

$$R_{max} = \frac{2V_{max} (L/D)_{max}}{C} \tan^{-1} \left( \frac{0.433 \frac{W_f}{W_i}}{1 - 0.25 \frac{W_f}{W_i}} \right)$$ (5.97)

In this equation, the angle in the arc tangent term must be expressed in radian. Using the same technique as presented in Section 5.4.2.3, the final lift coefficient as a function of the initial lift coefficient is derived as:

$$C_{L_2} = C_{L_1} \left( 1 - \frac{W_f}{W_i} \right)$$ (5.97a)

It is emphasized again that among three maximum ranges for three flight programs (equations 5.95, 5.96, 5.97), the second case (i.e., cruise-climb; equation 5.96) delivers the maximum range. The following example demonstrates this point.

**Example 5.6**

A jet transport aircraft with a mass of 100,000 kg, a fuel mass of 30,000 kg, and following features is flying at 30,000 ft altitude.

- \( S = 341.5 \text{ m}^2 \), \( C_{D_0} = 0.016 \), \( K = 0.065 \), \( C = 0.8 \text{ lb/hr/lb} \)

**a.** If this aircraft flies with the speed of 325.8 knot, how far it can fly without refueling? Determine the range for three following options:

1. Constant altitude-constant lift coefficient flight
2. Constant airspeed-constant lift coefficient flight
3. Constant altitude-constant airspeed flight

**b.** What is the velocity such that the aircraft delivers the maximum range?

**c.** Determine the maximum range for the same three options.

**Solution:**

**a. Regular range**

Using Appendix B, at the altitude of 30,000 ft, the air density ratio is 0.374. To find lift-to-drag ratio, we have:

$$C_L = \frac{2W}{\rho SV^2} = \frac{2 \times 100000 \times 9.81}{1.225 \times 0.374 \times 341.5 \times (325.8 \times 0.5144)^2} \Rightarrow C_L = 0.446$$ (2.3)

$$C_D = C_{D_0} + KC_L^2 = 0.016 + (0.065 \times 0.446^2) \Rightarrow C_D = 0.0289$$ (3.5)

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{0.446}{0.0289} = 15.4$$ (5.13)

The maximum lift-to-drag ratio is:

$$\left( \frac{C_L}{C_D} \right)_{max} = \frac{1}{2 \sqrt{KC_{D_a}}} = \frac{1}{2 \sqrt{0.065 \times 0.016}} = 15.5$$ (5.24)

Now we calculate range at three different cases:

1. Constant-altitude, constant-lift-coefficient flight
\[ R = \frac{2}{C \cdot D} V \left( 1 - \frac{1}{W_l} \right) = \frac{2 \times 15.4 \times 325.8}{0.8} \left[ 1 - \frac{30000 \times 9.81}{100000 \times 9.81} \right] \]

\[ \Rightarrow R = 2048 \text{ nm} = 3794 \text{ km} \]

(Note that the unit of \( V \) is nm/hr (i.e., knot) and the unit of \( C \) is 1 per hour.)

2. Constant-airspeed, constant-lift-coefficient flight

\[ R = \frac{V(L/D)}{C} \ln \left( \frac{1}{1 - \left( \frac{W_l}{W_i} \right)} \right) = \frac{325.8 \times 15.4}{0.8} \ln \left( \frac{1}{1 - 0.3} \right) \Rightarrow R = 2237 \text{ nm} = 4143 \text{ km} \]

3. Constant-altitude, constant-airspeed flight

\[ R = \frac{2V(C_L/C_D)_{\max} \tan^{-1} \left[ \frac{(C_L/C_D)_{\max} \frac{W_l}{W_i}}{2(C_L/C_D)_{\max} \left( 1 - KC_l \frac{C_L/C_D}{W_l/W_i} \right)} \right]}{C} \]

\[ R = \frac{2 \times 325.8 \times 15.5}{0.8} \left[ \frac{15.4 \times 0.3}{2 \times 15.5 \times (1 - 0.065 \times 0.446 \times 15.4 \times 0.3)} \right] \]

\[ \Rightarrow R = 2151.4 \text{ nm} = 3984 \text{ km} \]

Comparison of the range of three flight programs demonstrates that the range of cruise-climb flight is 4% more than the range of constant altitude-constant velocity flight; and 9% more than the range of constant-altitude, constant-lift-coefficient flight. However, these ranges are not necessarily maximum ranges.

b. Maximum range velocity

The maximum range velocity is independent of flight program, but is a function of initial flight altitude.

\[ C_{L_{\max}} = \sqrt{\frac{C_{D_s}}{3K}} = \sqrt{\frac{0.016}{3 \times 0.065}} = 0.286 \]

\[ V_{\max} = \sqrt{\frac{2mg}{\rho SC_{L_{\max}}}} = \sqrt{\frac{2 \times 100000 \times 9.81}{1.225 \times 0.374 \times 341.5 \times 0.286}} = 209.4 \frac{m}{\text{sec}} \]

\[ V_{\max} = 407.1 \text{ knot} = 0.69 \text{ Mach} \]

c. Maximum range

We substitute the maximum range velocity into equations 5.96, 5.94, and 5.95 to determine the maximum range in three flight programs:

1. Constant altitude-constant lift coefficient flight
The maximum range of three flight programs are different. The results show that the maximum range for cruise flight is 9% higher than the range with the speed of 325.8 knot. The results of this section is very similar to the results of section a. The question we would like to find the answer is that, as we carry more fuel in the tanks, what happens to difference between the maximum ranges of three flight programs.

5.4.5. Practical Considerations

5.4.5.1. Optimum Fuel Weight

One of the design applications of the range performance analysis is to determine the optimum fuel weight. It is evident that; an aircraft can fly longer with more fuel in tanks. In example 5.7, we saw that the ranges of three flight programs are different. The question we would like to find the answer is that, as we carry more fuel in the tanks, what happens to difference between the maximum ranges of three flight programs.

The ratio between the maximum range for cruise-climb flight to that for constant-altitude, constant-lift-coefficient flight is:

\[ \frac{R_{\text{max}2}}{R_{\text{max}1}} = \frac{0.866V_{\text{max}e}^2 (L/D)_{\text{max}}}{C} \ln \left( \frac{1}{1-G} \right) = \frac{\ln \left( \frac{1}{1-G} \right)}{2\left[1 - \sqrt{1-G} \right]} \]  

(5.97b)

Similarly, the ratio between the range for cruise-climb flight and the range for constant-altitude, constant-speed flight is:
\[ \frac{R_{\text{max} 2}}{R_{\text{max} 3}} = \frac{0.866V_{\text{max}, x} (L/D)_{\text{max}}}{C} \ln \left( \frac{1}{1 - G} \right) = \frac{0.433 \ln \left( \frac{1}{1 - G} \right)}{1 - 0.25G} \]

where, \( G \) denotes the ratio of fuel weight to aircraft weight; an sometime referred to as the fuel fraction.

\[ G = \frac{W_f}{W_1} \]  

\[ (5.97c) \]

Figure 5.23. Relative maximum range as a function of the fuel fraction

These relative ranges are plotted in figure 5.23. As the amount of fuel fraction is increased (\( G \) increased), the difference among the ranges for the three flight programs get wider. The figure indicates that differences in range among the various flight programs are not as important for short-range flight as for long-range flight.

The cruise-fuel weight fraction is a measure of the fuel available for cruise; all of the fuel loaded on the ramp is not available for cruise. A portion of the fuel is used for taxiing, take off, and climb to the cruise altitude. Moreover, there must be fuel remaining at the end of cruise for safety reasons. For instance; the wind speed and direction were not experienced as planned, or destination is not open for landing. In the latter case; under instrument flight rules (IFR); there must be sufficient fuel to fly to an alternate airport.

For a large transport aircraft, it is reasonable to assume that 3% of total fuel is consumed prior to the start of cruise and that the reserve is also 10% of fuel weight, leaving an 87% of total fuel available for cruise. For a GA aircraft, it may be assumed that 6% of total fuel (\( W_i \)) is consumed to reach cruise altitude, and the same reserve, so that the 84% of total fuel is available for cruise. With these assumptions, the initial cruise weight; \( W_1 \) is:

\[ W_1 = W_{f0} - 0.03W_f \]  

(large transport aircraft) \( (5.99a) \)

\[ W_1 = W_{f0} - 0.06W_f \]  

(small GA aircraft) \( (5.99b) \)
If the available fuel for cruise is not needed and not consumed, the performance of the aircraft will be penalized by carrying the fuel as dead weight.

5.4.5.2. Wind Effect
Wind, gust, disturbance are few permanent features of the atmosphere. The dynamics of flight is affected by atmospheric phenomena. The main influence of such atmospheric phenomena as gust and disturbance is on stability and control. However main influence of wind is on aircraft performance. To have a safe flight as defined by Federal Aviation Regulations, aircraft must be stable in the presence of gust and disturbance. A strong gust or disturbance may lead an aircraft to crash. Therefore atmospheric phenomena are significant factor in evaluation of aircraft performance. In this section, the effect of wind on the range is introduced.

The range is a function of ground speed which is affected by wind speed. The aerodynamic forces (e.g., lift and drag) are functions of relative airspeed, and not the wind speed. In effect, when an aircraft is experiencing a headwind, the lift and drag will not change, while the range will be reduced. Conversely, when an aircraft is experiencing a tailwind, the range will be increased. When an aircraft is cruising into a headwind, the airspeed is:

\[ V_G = V_{\infty} - V_w \]  

(5.100)

In contrast, when an aircraft is cruising with a tailwind, the airspeed is:

\[ V_G = V_{\infty} + V_w \]  

(5.101)

The range of an aircraft is strongly influenced by the wind. The prevailing direction of wind in the Northern Hemisphere is from South-west to North-east while in the Southern Hemisphere is from North-east to South-west. However because of the rotation of the earth, and its Coriolis Effect, the direction tilts by few degrees. The Coriolis Effect is the deflective effect of Earth’s rotation on all free-moving objects, including the atmosphere. The deflection is to the right in the Northern Hemisphere and to the left in the Southern Hemisphere. Furthermore the wind speed at high altitude is higher that wind speed at low altitude, due to the ground friction.

Depending on the direction of the aircraft, wind direction has an angle with aircraft direction (from zero to 360 degrees). Two important winds for straight level flight are those with zero (head-wind or nose-wind) and 180 degrees (cross-wind) direction. In these instances, the wind could be either headwind or tailwind. A headwind has an opposite direction compared to the aircraft direction, while a tailwind has the same direction. Consider a 30 knot wind that blows from East to West. This wind is a headwind for an aircraft that is flying from West to East. However, this wind is a tailwind for an aircraft that is cruising from East to West.

In chapter two the term airspeed was introduced. Aircraft airspeed is not affected by any wind, while wind affects the ground speed. A headwind decreases the aircraft ground speed, but a tailwind increases the aircraft speed relative to the ground. Since the range is calculated relative to the ground, the ground speed is employed instead of airspeed. This is the reason why the range is influenced by wind.

To calculate the range of an aircraft in the presence of the wind, the following equation is used:

\[ V_G = V_w \pm V_{\infty} \]  

(5.102)

where the \( V_\infty \) is the aircraft airspeed and \( V_w \) is the wind speed. The plus sign is used when aircraft faces and tailwind and minus sign is used when a headwind is present. Therefore a headwind will decrease the range, while a tailwind will increase the range. Since the wind affects the range, it will consequently influence the flight duration. Due to the three flight
operations in a cruising flight, we derived six equations for range so far; three equations for regular range, and three equations for maximum range. To include the effect of wind on range; in all of these six equations; just replace the \( V \) with \( V_G \) (i.e., \( V_w \pm \Delta V \)). For instance, the equations of range and maximum range, for the cruise-climb flight operation (i.e., constant velocity, constant-lift coefficient) are shown here:

\[
R_{w2} = \frac{(V_\infty \pm V_w)(L/D)}{C} \ln \left( \frac{1}{1 - \left( \frac{W_f}{W_i} \right)} \right) 
\]

\[
R_{max \, W} = R_{max \, W_2} = \frac{0.866(V_{max, W} \pm V_w)(L/D)_{max}}{C} \ln \left( \frac{1}{1 - \left( \frac{W_f}{W_i} \right)} \right) 
\]

where \( R_{w2} \) and \( R_{max \, W_2} \) indicate the range and maximum range respectively when the wind effect is included. Other parameters will not be varied.

Example 5.7

The jet transport aircraft DC-9-30 with 100 seats is cruising at 30,000 ft altitude with the speed of Mach 0.78. This aircraft that has a mass of 44,000 kg at the beginning of cruising flight is planned to descend and then land, when 7,000 kg of its fuel is consumed. Turbofan engines of this aircraft (JT8D-1S) have fuel specific consumption of 0.82 lb/hr/lb. Other features of this aircraft are:

\[
e = 0.82, \quad b = 29 \text{ m}, \quad C_{Do} = 0.02, \quad S = 93 \text{ m}^2
\]

a. What is the maximum range of this aircraft at this flight?
b. Determine the range of this flight (i.e. constant altitude, constant airspeed).
c. Determine the required thrust of this flight.
d. What is fuel consumption rate (in kg/hr)?
e. Determine the duration of this flight.
f. If the plan is to fly with maximum range objective, determine the altitude at the end of the flight.
g. What is the seat-km capacity of this aircraft per one liter of fuel?
h. Compare this feature of the aircraft with a car that consumes 10 liter of fuel per each 100 km.

Assume ISA condition.

Solution:

a. Maximum range

\[
AR = \frac{b^2}{S} = \frac{29^2}{93} = 9
\]

\[
K = \frac{1}{\pi e AR} = \frac{1}{3.14 \times 0.82 \times 9} = 0.043
\]

\[
C_{L, max} = \sqrt{\frac{C_{Do}}{3K}} = \sqrt{\frac{0.02}{3 \times 0.043}} = 0.394
\]
\[ V_{\text{max},e} = \sqrt{\frac{2mg}{\rho S C_D \sqrt{\frac{C_{D_{L}}}{3K}}}} = \sqrt{\frac{2 \times 44000 \times 9.81}{1.225 \times 0.374 \times 93 \times 0.394}} = 226.8 \text{ m/sec} \]  

(5.71)

At 30,000 ft, the speed of sound is 303 m/sec, so

\[ V_{\text{max},e}^\prime = \frac{226.8}{303} = 0.748 \text{ Mach} \]

\[ \left( \frac{C_L}{C_D} \right)_{\text{max}} = \frac{1}{2 \sqrt{KC_{D_{L}}}} = \frac{1}{2 \sqrt{0.043 \times 0.02}} = 17 \]  

(5.19)

\[ G = \frac{W_f}{W_i} = \frac{m_f}{m_i} = \frac{7000}{44000} = 0.159 \]  

(5.79)

\[ R_{\text{max}} = R_{\text{max},2} = \frac{0.866 V_{\text{max},e} (L/D)_{\text{max}}}{C} \ln \left( \frac{1}{1 - \frac{W_f}{W_i}} \right) \]

\[ = \frac{0.866 \times 17 \times 226.8 \times 3.6}{0.82} \ln \left( \frac{1}{1-0.159} \right) \Rightarrow R_{\text{max}} = 2538.4 \text{ km} \]

Note that the number 3.6 is used to convert C to 1/sec.

b. range at the specified flight condition:

\[ C_L = \frac{2W}{\rho S V^2} = \frac{2 \times 44000 \times 9.81}{1.225 \times 0.374 \times 93 \times (303 \times 0.78)^2} \Rightarrow C_L = 0.363 \]  

(2.3)

\[ C_D = C_{D_{L}} + KC_L^2 = 0.02 + \left(0.043 \times 0.363^2\right) \Rightarrow C_D = 0.0256 \]  

(3.11)

While the lift-to-drag ratio is:

\[ \frac{L}{D} = \frac{C_L}{C_D} = \frac{0.363}{0.0256} = 14.13 \]

\[ R = 2V \left( \frac{C_L}{C_D} \right)_{\text{max}} \tan^{-1} \left[ \frac{(C_L/C_D) \frac{W_f}{W_i}}{2(C_L/C_D)_{\text{max}} \left(1 - KC_L (C_L/C_D) \frac{W_f}{W_i}\right)} \right] \]

\[ R = \frac{2 \times 0.78 \times 303 \times 3.6 \times 17}{0.82} \tan^{-1} \left[ \frac{14.13 \times 0.159}{2 \times 17 \times [1-0.043 \times 0.363 \times 14.13 \times 0.159]} \right] \]

\[ R = 2413.18 \text{ km} \]

It is observed that this range is 125 km shorter than the maximum range.

c. \[ D = \frac{1}{2} \rho V^2 SC_D \]

\[ D = \frac{1}{2} \times 1.225 \times 0.374 \times (303 \times 0.78)^2 \times 93 \times 0.0256 \Rightarrow D = 30463 \quad N \approx 30 \text{ kN} \]
The engine thrust must be equal to aircraft drag, so engine must produce about 30 kN of thrust for this flight.

d. Flow rate:

\[ C = \frac{Q_f}{T} \Rightarrow Q_f = TC = \frac{30463 \times 0.82}{9.81} = 2546 \text{ kg hr} \]  \hspace{1cm} (5.55)

e. Flight duration

\[ t = \frac{m_f}{Q_f} = \frac{7000}{2546.3} = 2.75 \text{ hr} \]  \hspace{1cm} (5.54)

f. Maximum range is obtained if the aircraft flies with cruise-climb flight program. Thus the altitude at the end of this flight would be

\[ \sigma_2 = \sigma_1 (1 - G) = 0.374 \times (1 - 0.159) = 0.3145 \]  \hspace{1cm} (5.86)

Referring to appendix B, this density ratio belongs to 34,500 ft altitude. Since the aircraft was initially flying at 30,000 ft, the increase in altitude will be 4,500 ft.

g. Seat-kilometer

\[ \frac{\text{seat-km}}{\text{kg of fuel}} = \frac{100 \times 0.78 \times 303 \times 3.6}{2546.3} = 33.4 \]

h. Comparison with car

A car that can carry five passengers has fifty seat-km capacity \((5 \times 100 / 10 = 50)\), so the ratio of the car capacity to that of this aircraft is:

\[ \frac{50}{33.4} = 1.497 \approx 1.5 \]

This means that a car; compared with an aircraft; has about 50% more seat-km capacity for each kg of fuel.

5.4.6. Comparison and Conclusion

Table 5.5 presents a summary of range equations with corresponding flight variables. In this table, the word “Given” means that the pilot can select any acceptable speed (i.e., a speed from stall speed to maximum speed) he/she wishes. In addition, the word “Same” indicates that the final value is the same as the initial value.

<table>
<thead>
<tr>
<th>Flight Program</th>
<th>Range Type</th>
<th>Equation</th>
<th>Speed</th>
<th>Altitude</th>
<th>Lift coefficient</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Initial</td>
<td>Final</td>
<td>Initial</td>
<td>Final</td>
</tr>
<tr>
<td>1</td>
<td>Regular</td>
<td>5.66</td>
<td>Given</td>
<td>5.72</td>
<td>Given</td>
<td>Same</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>5.96</td>
<td>5.94</td>
<td>5.72</td>
<td>Given</td>
<td>Same</td>
</tr>
<tr>
<td>2</td>
<td>Regular</td>
<td>5.77</td>
<td>Given</td>
<td>Same</td>
<td>5.183</td>
<td>5.77b</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>5.94</td>
<td>5.90</td>
<td>5.90</td>
<td>Given</td>
<td>5.183</td>
</tr>
<tr>
<td>3</td>
<td>Regular</td>
<td>5.84</td>
<td>Given</td>
<td>Same</td>
<td>5.84b</td>
<td>5.84g</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>5.95</td>
<td>5.90</td>
<td>5.90</td>
<td>Given</td>
<td>Same</td>
</tr>
</tbody>
</table>

Table 5.5. A summary of range equations
5.5. Endurance

Another significant aircraft performance item and a design parameter is the endurance. For some aircraft, the most important performance parameter of the mission is to be airborne as long as possible (i.e., the longest endurance). Few instances are:

- An aircraft that is ready to land on destination airport, but the airport is not ready or busy. In this case, the pilot must loiter above the airport until the landing permission is issued by airport control tower.

- An anti-submarine airplane must wait airborne until they can identify an enemy submarine. In this case, the pilot must fly around a specific target area until a submarine is coming up.

- A reconnaissance aircraft must wait airborne in an area close to the target area until it can receive the enemy signal.

- An early warning aircraft (i.e., Airborne early warning and control (AWACS); e.g., Boeing E-3 Sentry) must hide airborne at a specific area to transmit commanded signals to its team aircraft.

- An aircraft that is planned to land on an airfield, but the weather at that area is not favorable for landing for a short time. In this case, the pilot must remain airborne until the weather cooperating and allows the aircraft to be landed.

- A border control aircraft must fly such that the crew can monitor the specific border area as long as possible.

- A relay aircraft is designed to receive a signal from a commander site and then to relay and transmit these signals after amplification to a final receiver (e.g., ground station).

At all of the above flight conditions, the best choice for a pilot is to be airborne with a speed such that it can stay in the air as long as possible. Many basic fundamentals for endurance and range are similar. The only difference is to consider how long (time) the aircraft can fly rather than how far (distance) it can fly. The objective for this flight is to minimize the fuel consumption, because the aircraft has limited fuel. A loiter is a flight condition that the endurance is its primary objective.

The Scaled Composites Virgin Atlantic GlobalFlyer [9] is an aircraft designed by Burt Rutan in which Steve Fossett flew a solo nonstop airplane flight around the world in a time of 67 hours 1 minute from February 28, 2005 until March 3, 2005 (world record). The flight speed of 319 knot broke the Absolute World Record for the fastest nonstop unrefueled circumnavigation set by the previous Voyager aircraft at 9 days 3 minutes and an average speed of 100 knot.

Another long endurance jet aircraft is Northrop Grumman RQ-4 Global Hawk [8]; an unmanned aerial vehicle (UAV) with a mass of 11,600 kg, a wing area of 50.2 m² is equipped with a turbofan engine with a maximum thrust of 31.4 kN. This surveillance aircraft has an endurance of 28 hours. In this section, the definition of endurance is presented, and then the technique to determine the maximum endurance velocity and maximum endurance is discussed.
5.5.1. The Definition of Endurance

Endurance (E) is the length of time that an aircraft can remain airborne for a given expenditure of fuel and for a specified set of flight condition. In effect, endurance includes all phases of flight such as take-off, climb, cruise, descent, and landing. However, due to simplicity, we only consider a level flight portion of the flight (i.e., cruise). For a long range transport aircraft, about 70-80 percent of the flight will be in cruising portion. For instance, a transport aircraft flying from New York City to Seattle spends about five hours in cruise, while about 20 minutes for take-off/climb, and 30 minutes for descent/landing. An interested reader may add the flight for other flight portion to calculate the total flight time.

In order to develop an expression for endurance, we employ the term “specific endurance (SE)” which is defined as the flight duration (t in hour/second) per unit of fuel mass (m in kg/slug) or fuel weight (W; in N/lb). For aircraft with turbofan or turbojet engine, the specific endurance is defined as:

\[
SE = \frac{dt}{-dW}
\]

The minus sign is used to account for a decrease in fuel weight during the flight operation. Recall, we are only considering a level flight, so the lift will be almost equal to aircraft weight (L = W), and engine thrust is approximately equal to aircraft drag (T = D).

By employing the definition of specific fuel consumption (C = Q/T); and fuel flow rate (Q = dW/dt), we can develop the following:

\[
\frac{dt}{dW} = \frac{dt}{Qdt} = \frac{1}{Q} = \frac{1}{CT} = \frac{1}{CD}
\]

Multiplying both numerator and denominator by lift (L) and replacing one of them with weight (W) results in:

\[
\frac{dt}{dW} = \frac{1}{CD} = \frac{L}{CDL} = \frac{L}{CDW} \Rightarrow dt = \frac{1}{CD} = -L \frac{dW}{CD W}
\]

In order to derive an equation for endurance, we now integrate the specific endurance for the entire time of cruising flight. It means from the time that aircraft has initial weight to when the aircraft has consumed its fuel (Wf).

\[
E = \int_{0}^{E} dt = \int_{W_1}^{W_2} \frac{-L dW}{CD W}
\]

where \(W_1\) is the aircraft weight at the beginning of the flight and \(W_2\) at the end of flight. Equation 5.108 is a general equation for endurance for a jet aircraft.

5.5.2. Endurance Calculation

Equation 5.108 is the general expression for the endurance of an aircraft. If the detailed variations of C, L, D, and W are known, one can numerically integrate to find an exact result for the endurance. The mathematical integration requires an implementation of practical considerations which will be discussed here. In order to solve the integration (Equation 5.108) and arrive to a closed-form solution, we need to set a few simplifying assumptions. During the duration of cruising flight (steady level), the lift should be kept almost equal (if the engine thrust contribution is neglected) to the aircraft weight, while the weight is gradually decreasing:

\[
L = W = \frac{1}{2} \rho SV^2 C_L
\]

At any weight, the speed is associated with an angle of attack, and in deed, with a lift coefficient (C_L) and an altitude. Assuming specific consumption is constant, and considering the equation
2.3 In mind, the equation 5.108 has independent parameters that are: weight (W), velocity (V), altitude or its corresponding air density (\( \rho \)), and angle of attack, or its associated lift coefficient (\( C_L \)). Since the fuel is consumed during flight, the aircraft weight is constantly decreased during the flight. In order to maintain a level flight, we have to decrease the lift as well. Of the many possible solutions only three are more practical and will be examined. In each case, two flight parameters will be held constant throughout cruise. The three options of interest during cruise are:

As it was discussed in the preceding section (5.4), there are three major flight programs to maintain the cruising flight for continuous decrease of the lift (Figure 5.20):

1. Decreasing flight speed (Constant-altitude, constant-lift coefficient flight)
2. Increasing altitude (Constant-airspeed, constant-lift coefficient flight)
3. Decreasing angle of attack (Constant-altitude, constant-airspeed flight)

For each flight program, the integral equation (5.62) will be set up and then only the final range equation will be shown and discussed. In the first option, the velocity must be reduced with the same rate as the aircraft weight is decreased. In the second solution, the air density must be decreased; in another word, the flight altitude must be increased. The third option offers the reduction of aircraft angle of attack; i.e., the reduction of lift coefficient. In terms of pilot operation, the first option is applied through throttle; and the third option is implemented through stick/yoke/wheel. In the second option, no action is needed by the pilot; the aircraft will gradually gain height (climbs). Based on the safety regulations and practical considerations, the second option is option of interest for majority of aircraft. The reason will be explained later. In general, when flight is conducted under the jurisdiction of Federal Aviation Regulations, the accepted flight program is the constant altitude-constant-airspeed flight program.

5.5.2.1. Flight program 1. Constant-altitude, constant-lift coefficient flight
In this flight program, the velocity is decreased as the weight is dropped. The drag coefficient is already defined as:

\[
C_D = C_{D_0} + KC_L^2
\]  
(3.11)

Plugging this equation into equation 5.108, and since \( L/D = C_L/C_D \) in a cruising flight, we can write:

\[
E = \frac{1}{C} \int \frac{dW}{W} C_{D_0} - \frac{1}{C} \int \frac{C_L}{C_{D_0} + KC_L^2} dW
\]  
(5.109)

The ratio of \( \frac{C_L}{C_{D_0} + KC_L^2} \) has a constant value, and will be taken out of the integration, so:

\[
E = \frac{1}{C} \int \frac{dW}{W} C_{D_0} + \frac{C_L}{C_{D_0} + KC_L^2} \int \frac{dW}{W}
\]  
(5.110)

The result of this integration is:

\[
E = \frac{C_L/C_D}{C} \ln \left[ \frac{W_1}{W_2} \right] = \frac{C_L/C_D}{C} \ln \left[ \frac{1}{1 - \frac{W_f}{W_1}} \right] = \frac{C_L/C_D}{C} \ln \left[ \frac{1}{1 - G} \right]
\]  
(5.111)
where $\ln \left[ \frac{W_1}{W_2} \right]$ is equal to $\ln[W_1] - \ln[W_2]$. The parameter $W_f$ denotes the fuel weight:

$$W_f = W_i - W_2$$  \hspace{1cm} (5.112)

Equation 5.111 implies that endurance is a function of lift-to-drag ratio, and fuel weight. It also indicates that the endurance is an inversely function of specific fuel consumption. As the fuel weight and the lift-to-drag ratio increase, the endurance will be increased. Furthermore, as the specific fuel consumption is decreased, the endurance will be improved.

It can be seen from equation 2.3 that the airspeed must be decreased as fuel is consumed along the flight path. To determine the final airspeed $V_2$; we know that the aircraft weight is equal to the lift at the beginning and at the end of flight. As we derived in section 5.4, this leads to the fact that the velocity at the end of flight is reduced to:

$$V_2 = V_i \sqrt{1 - \frac{W_f}{W_i}}$$  \hspace{1cm} (5.113)

Since $C_L$ is held constant, equations 5.5, 5.6, and 3.1 implies that the thrust must be constantly decreased (by constantly setting back the throttle) as the fuel is used (i.e., the gross weight decreases).

5.5.2.2. Flight program 2. Constant-airspeed, constant-lift coefficient flight

In this flight program, the altitude is increased as the aircraft weight is dropped (i.e., cruise-climb). By inspection, we notice that this flight option results in the same closed-form solution as the first flight option:

$$E_2 = \frac{C_L}{C_D} \frac{1}{C} \ln \left[ \frac{1}{1-G} \right]$$  \hspace{1cm} (5.114)

By inspection, we can conclude that:

$$E_2 = E_1$$  \hspace{1cm} (5.115)

Two flight programs of 1 and 2 have a similar endurance.

The cruise altitude for a cruise-climb flight option is gradually increasing. The altitude at the end of cruise-climb flight ($h_2$) can be expressed in terms of both the initial altitude ($h_1$) and the fuel fraction. The derivation is in Section 5.7. The density ratio at the end of cruise-climb flight ($\sigma_2$) in terms of both the initial density ratio and the fuel fraction is:

$$\sigma_2 = \sigma_1 (1 - G)$$  \hspace{1cm} (5.116)

where, $\sigma_1$ is air density ratio at the beginning of the cruise, and $G$ is the fuel fraction. The climb angle is so small that can be ignored. When the air density at the end of cruise-climb flight ($\sigma_2$) obtained, one can utilize Appendix A or B to determine the final altitude.

5.5.2.3. Flight program 3. Constant-altitude, constant-airspeed flight

In this flight program, the lift coefficient is decreased as the aircraft weight is dropped. In effect, the angle-of-attack will be reduced via stick/wheel as the aircraft weight is decreased. Since the lift is equal to the weight during cruise, the $L$ from numerator and $W$ from denominator of equation 5.108 are eliminated. Thus the endurance equation for this flight program can be simplified as:

$$E = - \frac{1}{C} \int_{W_i}^{W_f} \frac{dW}{D}$$  \hspace{1cm} (5.117)
The variations of drag with respect to aircraft weight in nonlinear. As derived in Chapter 5 (Equation 5.30), the drag is a function of aircraft weight as follows:

\[ D = \frac{1}{2} \rho V^2 S C_{D_o} + \frac{2KW^2}{\rho V^2 S} \] 

(5.30)

Plugging the equation 5.30 into equation 5.117 yields:

\[ E_3 = -\frac{1}{C} \int_{w_1}^{w_2} dW \left( \frac{1}{2} \rho V^2 S C_{D_o} + \frac{2KW^2}{\rho V^2 S} \right) \] 

or

\[ E_3 = -\frac{1}{C} \left( \frac{2K}{\rho V^2 S} \right) \int_{w_1}^{w_2} dW \left( \frac{\rho V^2 S}{2C_{D_o}} + W^2 \right) \]

(5.118)

Using a mathematical reference [9], the solution of the integration with a similar form is determined:

\[ \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \] 

(5.120a)

where

\[ a = \frac{1}{2} \rho V^2 S \left( \frac{C_{D_o}}{K} \right)^{\frac{1}{2}} \] 

(5.120b)

Therefore, the solution for the integration is:

\[ E_3 = -\frac{1}{C} \left( \frac{2K}{\rho V^2 S} \right) \frac{1}{2} \rho V^2 S \left( \frac{C_{D_o}}{K} \right)^{\frac{1}{2}} \left[ \tan^{-1} \left( \frac{W}{\frac{1}{2} \rho V^2 S \left( \frac{C_{D_o}}{K} \right)^{\frac{1}{2}}} \right) \right] \] 

(5.121)

Since the maximum lift-to-drift ratio is related to \( \sqrt{KC_{D_o}} \), as \( \sqrt{KC_{D_o}} = \frac{1}{2(C_L/C_D)_{\text{max}}} \), the equation will be simplified to:

\[ E_3 = \frac{2(L/D)_{\text{max}}}{C} \left[ \tan^{-1} \left( \frac{2W_1}{\rho V^2 S \sqrt{C_{D_o}/K}} \right) - \tan^{-1} \left( \frac{2W_2}{\rho V^2 S \sqrt{C_{D_o}/K}} \right) \right] \] 

(5.122)

As derived in Chapter 5 (Equation 5.40), the term \( \sqrt{C_{D_o}/K} \) is equal to the minimum-drag lift coefficient; \( C_{\text{min,D}} \).

\[ E_3 = \frac{2(L/D)_{\text{max}}}{C} \left[ \tan^{-1} \left( \frac{2W_1}{\rho V^2 S C_{\text{min,D}}} \right) - \tan^{-1} \left( \frac{2W_2}{\rho V^2 S C_{\text{min,D}}} \right) \right] \] 

(5.123)

The bracketed term represents the difference between two angles (in radian). The equation 6.83 is valid for any given constant altitude, and any given constant velocity, as long as the altitude and the velocity are permissible (i.e., within the flight envelope of the aircraft). In this flight
program, the initial lift coefficient, and the final lift coefficient are readily obtained by equations 5.84b and 5.84c respectively. Inserting the relationship between initial weight, final weight, and fuel weight \( (\text{W}_f = \text{W}_i (1 - \text{G})) \) into equation 6.84, and converting the difference between the tangent of two angles, we may reformat the equation 5.122 into:

\[
E_3 = \frac{2(C_L/C_D)_{\text{max}}}{C} \tan^{-1} \left[ \frac{(C_L/C_D)_h \text{G}}{2(C_L/C_D)_{\text{max}} (1 - KC_{\text{L}}^2/C_D) \text{G}} \right]
\]

(5.124)

where \( C_{\text{L}} \) is the initial lift coefficient (equation 5.84b), and \( (C_L/C_D)_h \) denotes the initial lift-to-drag ratio (i.e., \( C_{\text{L}} \left/ \left( C_{\text{D}} + KC_{\text{L}}^2 \right) \right\) ). From the theoretical point of view, all three flight programs are realizable, but in practice; only the third case is acceptable, and approved for GA aircraft by FAA. The third case is hard to follow by a human pilot, because the pilot must constantly decrease the angle of attack, through deflection of the elevator. However, for an aircraft equipped with an autopilot; this is an easy task. This case is the safest flight program among three possible programs.

<table>
<thead>
<tr>
<th>No</th>
<th>Aircraft</th>
<th>Type</th>
<th>T (kN)</th>
<th>Mass (kg)</th>
<th>Endurance (hr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>British Aerospace Harrier II</td>
<td>Close support</td>
<td>105.9</td>
<td>14,061</td>
<td>3 hr</td>
</tr>
<tr>
<td>2</td>
<td>BAE Hawk 60</td>
<td>Attack</td>
<td>25.35</td>
<td>5,700</td>
<td>2 hr, 42 min</td>
</tr>
<tr>
<td>3</td>
<td>Boeing E-3 Sentry</td>
<td>Airborne warning</td>
<td>2x273.6</td>
<td>171,255</td>
<td>10 hr</td>
</tr>
<tr>
<td>4</td>
<td>McDonnell Douglas F/A-18 Hornet</td>
<td>Fighter</td>
<td>2x71.2</td>
<td>25,401</td>
<td>1 hr, 45 min</td>
</tr>
<tr>
<td>5</td>
<td>Northrop Grumman E-8 Joint- STARS</td>
<td>Early warning</td>
<td>4x80.1</td>
<td>152,407</td>
<td>11 hr</td>
</tr>
<tr>
<td>6</td>
<td>Northrop Grumman RQ-4 Global Hawk</td>
<td>Reconnaissance</td>
<td>31.4</td>
<td>11,600</td>
<td>28 hr</td>
</tr>
<tr>
<td>7</td>
<td>Aeromacchi MB-339C</td>
<td>Fighter</td>
<td>19.57</td>
<td>6,350</td>
<td>3 hr, 50 min</td>
</tr>
<tr>
<td>8</td>
<td>Avioane IAR 99</td>
<td>Ground attack</td>
<td>17.79</td>
<td>4,400</td>
<td>2 hr, 40 min</td>
</tr>
<tr>
<td>9</td>
<td>Mikoyan MiG-31</td>
<td>Interceptor</td>
<td>2x91.3</td>
<td>41,000</td>
<td>3 hr, 36 min</td>
</tr>
<tr>
<td>10</td>
<td>Lockheed U-2</td>
<td>Reconnaissance</td>
<td>76</td>
<td>18,600</td>
<td>&gt;10 hr</td>
</tr>
<tr>
<td>11</td>
<td>Virgin Atlantic Global Flyer</td>
<td>nonstop flight around the world</td>
<td>10</td>
<td>10,024</td>
<td>67</td>
</tr>
<tr>
<td>2</td>
<td>Dassault/Dornier Alpha Jet</td>
<td>Trainer</td>
<td>8000</td>
<td>2x14.1 kN</td>
<td>3, 30'</td>
</tr>
<tr>
<td>3</td>
<td>Panavia Tornado</td>
<td>Fighter</td>
<td>24500</td>
<td>2x71.2 kN</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>Saai-marchetti S.211</td>
<td>Fighter</td>
<td>2750</td>
<td>11.12 kN</td>
<td>3, 50'</td>
</tr>
<tr>
<td>5</td>
<td>BAE Hawk 60</td>
<td>Trainer</td>
<td>8570</td>
<td>23.75 kN</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>BAE Harrier</td>
<td>Interceptor</td>
<td>11880</td>
<td>95.6 kN</td>
<td>1, 30'</td>
</tr>
<tr>
<td>7</td>
<td>Boeing E3 Sentry</td>
<td>Early warning</td>
<td>147417</td>
<td>4x93.4 kN</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>Boeing E-6A</td>
<td>Relay</td>
<td>155128</td>
<td>4x97.9 kN</td>
<td>15, 24'</td>
</tr>
<tr>
<td>14</td>
<td>Manchang L-8</td>
<td>Trainer</td>
<td>4200</td>
<td>15.57 kN</td>
<td>4, 30'</td>
</tr>
<tr>
<td>18</td>
<td>Mirage F-1</td>
<td>Fighter</td>
<td>16200</td>
<td>70.6 kN</td>
<td>2, 15'</td>
</tr>
<tr>
<td>19</td>
<td>Breguet 1150 Atlantic</td>
<td>Air Patrol</td>
<td>46200</td>
<td>2x1600 hp</td>
<td>18</td>
</tr>
</tbody>
</table>

**Table 5.6. Endurance of several jet aircraft**

A comparison between endurance equations (5.111, 5.114, and 5.124) and range equations (e.g., equation 5.77) shows that endurance is simply equal to range divided by airspeed. Three endurance equations \((E_1, 5.111, E_2, 5.114, \text{and} E_3, 5.124)\) provide the technique to find the
endurance for a jet aircraft while at a cruising flight with given flight conditions. In endurance equations, the unit of endurance is reciprocal of the unit of specific fuel consumption (C). For instance, if the unit of C is 1/hr, the unit of endurance will be in terms of hour.

5.5.3. Maximum Endurance Velocity

As we classified earlier (section 5.4), there are three following options in a cruising flight for an endurance mission: 1. Constant altitude-constant lift coefficient flight; 2. Constant airspeed-constant lift coefficient flight; and 3. Constant altitude-constant airspeed flight. These can be regrouped into two groups: 1. Constant speed flight (case 2 and 3); 2. Non constant-speed flight (case 1). For the cases 2 and 3, the technique to determine the maximum endurance velocity (V_{maxE}) to maximize the endurance is presented.

In this section, the technique to determine the maximum endurance velocity (V_{maxE}) to maximize the endurance for the cases 2 and 3 (i.e., constant-speed) is presented. For a non-VTOL\(^1\) aircraft (such as fixed-wing aircraft Harrier, or rotary wing aircraft, Bell 206), it is impossible to fly with zero velocity (in fact, it is impossible to fly with a speed less than stall speed). Thus, in order to maximize the flight duration, the aircraft must fly with a specific speed that results in the maximum endurance.

In a jet aircraft the specific fuel consumption is almost constant with speed, so in order to minimize the fuel consumption, the aircraft should fly such that it requires minimum thrust. This implies an aircraft cruising airspeed that produces the minimum drag. Furthermore, equation 5.114 confirms that in order to maximize the endurance, the aircraft must fly with a speed such that that the (L/D)_{max} is maximized. Therefore maximum endurance speed is the same as minimum drag speed as follows:

\[
V_{maxE} = V_{minD} = \frac{2mg}{\rho S \sqrt{C_{D_o} \frac{C_{L}}{K}}}
\]

We note from equation 5.125 that the maximum endurance velocity is a function of aircraft weight and altitude, and inverse function of wing area and zero lift drag coefficient. As noted, the maximum endurance velocity is proportional to the square root of air density (i.e., altitude); it increases with altitude.

By comparing equations 5.125 and 5.39, we can conclude that the lift coefficient at maximum endurance speed is equal to square root of ratio of the zero lift drag coefficient (C_{D_o}) and induced drag factor (K). Hence

\[
C_{L_{maxE}} = C_{L_{maxD}} = C_{L_{(L/D)_{max}}} = \sqrt{\frac{C_{D_o}}{K}}
\]

The typical value for the maximum endurance lift coefficient (C_{L_{maxE}}) is about 0.3-0.6.

In theory, the maximum endurance velocity may be lower than the minimum drag speed (V_{maxE} < V_{minD}); or could be higher than the minimum drag speed (V_{maxE} > V_{minD}). However, in practice, only a maximum endurance velocity higher than the minimum drag speed is allowable. As discussed in section 5.3.2, when the maximum endurance velocity is lower than the minimum drag speed, the aircraft will be in an unstable condition. So, it is impractical to fly with minimum drag speed for majority of aircraft. Moreover, in theory, the maximum

\(^1\) Vertical take-Off and Landing
endurance velocity may be lower than the stall speed \( V_{maxE} < V_s \); or could be higher than the stall speed \( V_{maxE} > V_s \). However, in practice, only a maximum endurance velocity higher than the stall speed is allowable.

For an aircraft that the maximum endurance speed is lower than the stall speed, a safe maximum endurance speed is selected (considered) to be about 10 to 20 percent higher than stall speed.

\[ V_{maxE} = k V_s \]  
\[ 1.1 < k < 1.2 \]

It should be emphasized that this velocity has been derived based on the assumption that the specific fuel consumption is constant with speed.

For a non-constant-speed flight operation (case 1), the velocity at the beginning of flight to maximize the endurance is assumed equal to the maximum endurance velocity, as derived before:

\[ V_1 = V_{maxE} = \sqrt{\frac{2mg}{\rho S \left( \frac{C_{L}}{C_{D}} \right)}} \]  
\[ V_{2} = V_1 \sqrt{1 - \frac{W_f}{W_i}} \]

Since \( C_L \) is held constant, equation 5.8 implies that the thrust must be constantly decreased (by constantly setting back the throttle) as the fuel is used (i.e., the gross weight decreases).

### 5.5.4. Maximum Endurance

Three endurance equations \( E_1, 5.111, E_2, 5.114, \) and \( E_3, 5.122 \) provide the technique to find the regular endurance for a jet aircraft. However, we are more interested in determining the maximum endurance for a jet aircraft. In this section, the technique to determine the maximum endurance \( E_{max} \) for a jet aircraft is developed.

As we classified earlier (section 5.4), there are three following options in a cruising flight for an endurance mission: 1. Constant altitude-constant lift coefficient flight; 2. Constant airspeed-constant lift coefficient flight; and 3. Constant altitude-constant airspeed flight. In Section 5.5.2, three equations were derived for calculations of regular endurance at any flight condition. Moreover, in section 5.3.3.1, three equations were developed for the speed to maximize velocity. Equations 5.111, 5.114 and 5.122 illustrate that in order to maximize the endurance during aircraft design phase; the specific fuel consumption must be minimized, while the fuel weight should be maximized (carry the highest possible of fuel). When an aircraft is manufactured, pilot can maximize endurance with selecting flight conditions. In the derivation process in this section, we assume that the specific fuel consumption \( C \) is constant with speed and altitude. Table 5.6 presents endurance of several jet aircraft.

### 5.5.4.1. Constant altitude-constant lift coefficient flight

In this flight program, the velocity is decreased as the weight is dropped. In section 5.3.3.1, the equation to calculate the regular endurance was developed (Equation 5.111). By inspection of equation 5.111, we notice that the endurance will be maximized when lift-to-drag ratio is at its maximum value (equation 5.19). Therefore, the maximum endurance is:
\[ E_{\text{max},1} = \frac{(C_L/C_D)_{\text{max}}}{C} \ln \left[ \frac{1}{1 - \frac{W_f}{W_i}} \right] \]  
(5.131)

This relationship implies that in order to maximize the endurance (in this flight program), the pilot must always cruise with any combination of lift coefficient, velocity, and altitude such that the flight delivers the maximum lift-to-drag ratio; \((L/D)_{\text{max}}\), this implies that the lift coefficient should always be:

\[ C_{L_{\text{max}}} = \sqrt{\frac{C_D}{K}} \]  
(5.132)

Since, the velocity is reduced during this flight program, the initial velocity will be:

\[ V_1 = \sqrt{\frac{2W_i}{\rho S \sqrt{\frac{C_D}{K}}}} \]  
(5.133)

Similarly, the final velocity will be:

\[ V_2 = \sqrt{\frac{2W_f}{\rho S \sqrt{\frac{C_D}{K}}}} \]  
(5.134)

In theory, this maximum endurance can be achieved at any fixed altitude; as long as the flight conditions are practical.

### 5.5.4.2. Constant airspeed-constant lift coefficient flight

In this flight program, the altitude is increased as the aircraft weight is dropped (i.e., cruise-climb). In section 5.3.3.1, the equation to calculate the regular endurance was developed (Equation 5.114). By inspection of equation 5.114, we notice that the endurance will be maximized when lift-to-drag ratio is at its maximum value (equation 5.19). Therefore, the maximum endurance is:

\[ E_{\text{max},2} = \frac{(L/D)_{\text{max}}}{C} \ln \left[ \frac{1}{1 - \bar{G}} \right] \]  
(5.135)

This relationship implies that in order to maximize the endurance (in this flight program), the pilot must always cruise with the velocity such that the flight delivers the maximum lift-to-drag ratio; \((L/D)_{\text{max}}\) (i.e., \(V_{\text{max},e}\); Equation 5.127) and a lift coefficient that has a value equal to \(\sqrt{\frac{C_D}{K}}\) (Equation 5.128). The only choice the pilot has in to select the initial altitude, and increase it along the flight.

As it was discussed in section 5.3.3, flight at the maximum endurance speed is often not realizable. The reason is the speed instability. To resolve this problem, most aircraft choose to fly at a speed that is 10 to 20 percent higher than the minimum drag speed. This leads in the endurance lower than the calculated one.

### 5.5.4.3. Constant altitude-constant airspeed flight.

In this flight program, the lift coefficient gradually decreases as the aircraft weight is dropped. In section 5.3.3.1, the equation to calculate the regular endurance was developed (Equation
5.124). By inspection of equation 5.124, we notice that, the maximum endurance is when the ratio inside the parenthesis is maximized:

\[
E_{\text{max}3} = \frac{2(C_L/C_D)_{\text{max}}}{C} \tan^{-1} \left[ \frac{(C_L/C_D)_{\text{max}} W_f}{W_1} \right] 
\]

By plugging equations 5.24, and 5.128 into equation 5.136, we obtain:

\[
E_{\text{max}3} = \frac{2(C_L/C_D)_{\text{max}}}{C} \tan^{-1} \left[ \frac{G}{2 \left( 1 - K \sqrt{\frac{C_{L_e}}{K} \frac{1}{2 \sqrt{K C_{D_e}}}} \right)} \right] 
\]

which results in:

\[
E_{\text{max}3} = \frac{2(L/D)_{\text{max}}}{C} \tan^{-1} \left[ \frac{0.5G}{1 - 0.5G} \right] 
\]

This relationship implies that in order to maximize the endurance (in this flight program), the pilot must always cruise with an initial velocity such that the flight delivers the maximum lift-to-drag ratio; \((L/D)_{\text{max}}\) (i.e., \(V_{\text{max},e}\); Equation 5.127) and an initial lift coefficient that has a value equal to \(\sqrt{\frac{C_{D_e}}{K}}\) (Equation 5.128). The only choice the pilot has is to select a fixed altitude, and to keep it along the flight. In equation 5.138, the initial lift coefficient is:

\[
C_{L_i} = \sqrt{\frac{C_{D_e}}{K}} 
\]

However, the final lift coefficient decreases to:

\[
C_{L_e} = C_{L_i} \frac{W_2}{W_1} 
\]

where \(W_2\) is the final weight at the end of cruise.

**Example 5.8**

Determine the maximum endurance for the transport aircraft DC-9-30 for a constant airspeed-constant lift coefficient flight program. The characteristics of this aircraft are given in example 5.8. Then, determine the maximum endurance speed if the aircraft cruises at sea level.

**Solution:**

Using the result of example 5.7, we have \((L/D)_{\text{max}} = 17\) and \(G = 0.159\)

Substitution, yields:

\[
E_{\text{max}} = \frac{(L/D)_{\text{max}}}{C} \ln \left[ \frac{1}{1 - G} \right] = \frac{17}{0.82} \ln \left[ \frac{1}{1 - 0.159} \right] \Rightarrow E_{\text{max}} = 3.59 \text{ hr} 
\]

Comparison of this value with the result of section “e” of example 5.7 reveals that if the pilot flies for maximum range, the flight duration is 3 hours, which is 0.59 hour shorter than this case.
To calculate the maximum endurance speed,

\[
C_{\text{max}e} = \sqrt[3]{\frac{C_{\text{D},e}}{K}} = \sqrt[3]{\frac{0.02}{0.043}} = 0.682
\]  

\[
V_{\text{max}e} = \frac{2mg}{\rho S \sqrt{C_{\text{D},e} / K}} = \frac{\sqrt{2 \times 4400 \times 9.81}}{1.225 \times 0.374 \times 93 \times 0.682} = 172.4 \text{ m/sec} = 0.57 \text{ Mach}
\]  

Note that the maximum range speed was Mach 0.748. This means that the maximum endurance is about 20% longer than a flight for maximum range. Also note that maximum endurance velocity is about 24% faster than the speed for the maximum range.

5.5.5. Practical Considerations

In this section, three topics pertaining practical considerations on endurance will be discussed briefly: 1. Altitude for maximum endurance, 2. Comparison between time to maximize range and the maximum endurance 3. Comparison between \(V_{\text{max}E}\) and \(V_{\text{max}R}\), and 4. Effect of wind on endurance.

5.5.5.1. Altitude for Maximum Endurance

In section 5.3.3, we derived three equations for maximum endurance (5.131, 5.135, and 5.138). Inspection of these equations indicates that the maximum endurance is independent of altitude; assuming specific fuel condition is constant with altitude and velocity. An aircraft cruising at high cruise altitude will have the same endurance as if cruise at sea level; provided that each flight has an appropriate maximum endurance speed. This is due to the fact that the maximum endurance speed is faster than the maximum endurance speed at sea level. However, flight at higher altitude yields a longer range.

The equations 5.131, 5.135, and 5.138 also show that the maximum endurance is independent of wing loading (W/S). This implies that a long endurance may be achieved by a large and heavy aircraft; or by a small and light aircraft, provided that the other flight conditions (e.g., velocity) are appropriate.

Please note, in the derivation process in this section, we assumed that the specific fuel consumption (C) is constant with speed and altitude. However, the specific fuel consumption varies with airspeed and flight altitude. For a jet aircraft equipped with a turbofan engine, the specific fuel consumption decreases with altitude, but increases with Mach number (See Figures 4.54 and 4.57). When this reality is included in our analysis, we can conclude that the maximum endurance will slightly increase with altitude, but slightly decreases with Mach number.

5.5.5.2. Comparison between \(t_{\text{max}R}\) and \(E_{\text{max}}\)

In example 5.7, we observed that flight duration of a mission to maximize the endurance is longer than that of maximum range. The difference between these two durations depends on the ratio of their fuel fractions. To have a better idea, we resort to mathematical operation. For instance, consider a constant-altitude, constant-speed flight program (case 3). Equation 5.138 demonstrates the expression for the maximum endurance, and equation 5.95 has the relationship for the maximum range. Both equations are repeated here for convenience:
\[
E_{\text{max}} = \frac{2(L/D)_{\text{max}} \tan^{-1} \left( \frac{0.5G}{1 - 0.5G} \right)}{C} 
\]

(5.138)

\[
R_{\text{max}} = \frac{2V_{\text{max}} (L/D)_{\text{max}} \tan^{-1} \left( \frac{0.433G}{1 - 0.25G} \right)}{C} 
\]

(5.95)

The duration of flight when maximizing range is just equal to the maximum range divided by velocity for maximum range:

\[
t_{\text{max}} \times \frac{R_{\text{max}}}{V_{\text{max}}} = \frac{2(L/D)_{\text{max}} \tan^{-1} \left( \frac{0.433G}{1 - 0.25G} \right)}{C} 
\]

(5.141)

Figure 5.24. Comparison between \(t_{\text{max}}\) and \(E_{\text{max}}\) for a constant-altitude, constant-speed flight program

Figure 5.25. Comparison between \(V_{\text{max}}\) and \(V_{\text{max}}\)

If we divide two durations (equation 5.138 and 5.141), the following is obtained:
The maximum endurance speed is the same as the minimum drag speed. However, the maximum range speed is determined by passing an asymptote to drag-speed curve (as shown in figure 5.25).

5.5.5.4. Effect of Wind on Endurance
Wind, gust, disturbance are few permanent features of the atmosphere. The dynamics of flight is affected by atmospheric phenomena. The prevailing direction of wind in the Northern Hemisphere is from South-west to North-east while in the Southern Hemisphere is from North-east to South-west. Furthermore the wind speed at high altitude is higher that wind speed at low altitude, due to the ground friction.

This relationship illustrates that the more fuel fraction, the larger the difference is between maximum endurance and the duration for maximum range. The maximum endurance is at least 15.5 percent longer than the duration for maximum range. At a fuel weight fraction of 50 percent, this gap is 34.8 percent (See figure 5.24).

\[
\frac{E_{\text{max}}}{t_{\text{max}, E_{\text{const}}}} = \frac{\tan^{-1}[0.5G/(1-0.5G)]}{\tan^{-1}[0.433G/(1-0.25G)]} \geq \frac{1.155 \times (1-0.25G)}{1-0.5G} \quad (5.142)
\]

5.5.5.3. Comparison between \(V_{\text{max}E}\) and \(V_{\text{max}R}\)
It is interesting and beneficial to compare the speed to maximize endurance with the speed to maximize range. To compare the maximum endurance speed with maximum range speed, equations 5.125 and 5.90 are used. Figure 5.25 shows these two speeds and their relationship with drag. In a jet aircraft, the maximum range speed is always higher than the maximum endurance speed.

The maximum endurance speed is the same as the minimum drag speed. However, the maximum range speed is determined by passing an asymptote to drag-speed curve (as shown in figure 5.25).

**Figure 5.26. Variations of engine thrust and aircraft drag with altitude**
In chapter two the term airspeed was introduced. Aircraft airspeed is not affected by any wind, while wind affects the ground speed. A headwind decreases the aircraft ground speed, but a tailwind increases the aircraft speed relative to the ground. The aerodynamic forces (e.g., lift and drag) are functions of relative airspeed, and not the wind speed. In effect, when an aircraft is experiencing a headwind, the lift and drag will not change, while the range will be reduced. Conversely, when an aircraft is experiencing a tailwind, the range will be increased.

The endurance is not a function of ground speed which is affected by wind speed. Wind does not affect airspeed or endurance, while it does affect the flight duration to fly over a fixed distance. The wind does not change the aircraft endurance. The reason is that the endurance is calculated based on the airspeed rather than ground speed. An aircraft with endurance of say 10 hours will have the same endurance in the presence of a 30 knot tailwind or headwind. With the same token, the maximum endurance of an aircraft will be the same with or without the presence of any wind.

5.6. Ceiling

5.6.1. Definition

Another very important criterion for aircraft performance is the ceiling. Ceiling is defined as the highest altitude that an aircraft can safely have a straight level flight. Another definition is the highest altitude that an aircraft can reach by its own engine and have sustained flight. The higher the ceiling, the better is the aircraft performance. This performance parameter has limited application in civil airplanes, but is very significant for military airplanes. For instance, if the ceiling of a fighter is higher than the ceiling of missiles in a specific region, this fighter can operate freely on that region and survive. The aircraft parameters that positively influence its ceiling are the lower weight, higher engine thrust, and lower drag. The materials in this section do not apply to an aircraft with rocket engine, since there is no limit for their ceilings.

The maximum ceiling of today’s aircraft is about 120,000 ft. This record belongs to a Soviet MIG-25 that could fly at 123,523 ft in 1987. The X-15A-3 experimental aircraft could reach the altitude of 314,750 ft after being launched from another aircraft in 1962. The primary reason for having ceiling is the air density. At high altitude, there is not sufficient air to be consumed by aircraft engine for combustion; hence the jet engine thrust drops with altitude. On the other hand, as the air density is decreased, the drag force is decreased too. But the rate of decrease in thrust is higher than the rate of decrease in drag (as it is shown by figure 5.26). These two curves have an intersection that is the altitude for ceiling.

As an aircraft flies higher and higher, the amount of available air is decreasing, so the available thrust is decreased too. As a result, on one altitude, the maximum available thrust is barely enough for aircraft to maintain its level flight. This is the very ceiling. This is true up to about 120,000 ft. Rockets have solved this problem by carrying their own air in a tank as well as fuel. Thus rockets and missiles with rocket engines do not have any limit in their ceiling. Any fighter that has a ceiling higher than the ceiling of enemy missile in a target area can survive. Otherwise, the fighter must rely on its maneuverability in order to operate on a target airspace.

In general there are five types of ceiling:
1. Absolute Ceiling ($h_{ac}$): As the name implies, absolute ceiling is the absolute maximum altitude that an aircraft can ever maintain level flight. In another term, the absolute ceiling is the altitude at which the rate of climb\(^2\) is zero. So the aircraft is not able to climb higher than the absolute ceiling. The absolute ceiling is sometime referred to as the maximum operating altitude.

2. Service Ceiling ($h_{sc}$): Service ceiling is defined as the highest altitude at which the aircraft can climb with the rate of 100 ft per minute (i.e., 0.5 m/sec). Service ceiling is lower than absolute ceiling.

3. Cruise Ceiling ($h_{cc}$): Cruise ceiling is defined as the highest altitude at which the aircraft can climb with the rate of 300 ft per minute (i.e., 1.5 m/sec). Cruise ceiling is lower than service ceiling.

\(^2\) Rate of climb is covered in Chapter 7.
4. Combat Ceiling ($h_{cc}$): Combat ceiling is defined as the highest altitude at which a fighter can climb with the rate of 500 ft per minute (i.e., 2.5 m/sec). Combat ceiling is lower than cruise ceiling. This ceiling is defined only for fighter aircraft.

<table>
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<th>No</th>
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<th>Type</th>
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<th>Mass (kg)</th>
<th>Service Ceiling (ft)</th>
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<td>Reconnaissance</td>
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<td>Russia</td>
<td>Close air support</td>
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<td>45,000</td>
<td>65,620</td>
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<td>Mikoyan MiG-25</td>
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<td>Interceptor</td>
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<td>India</td>
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<td>Fighter</td>
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<td>15.2</td>
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<td>Boeing B-52 Stratofortress</td>
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<td>2×105.7</td>
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<td>Lockheed SR-71 Blackbird</td>
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<td>Reconnaissance</td>
<td>2×151.3</td>
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<td>Lockheed U-2C</td>
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<td>Reconnaissance</td>
<td>75.6</td>
<td>7835</td>
<td>90,000</td>
</tr>
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<td>Dassault Falcon 2000</td>
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<td>Boeing 747-400</td>
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<td>Airliner</td>
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<td>Regional jet airliner</td>
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<td>37,000</td>
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<td>27</td>
<td>F-22 Raptor</td>
<td>Lockheed Martin</td>
<td>Stealth Fighter</td>
<td>2×155</td>
<td>39,000</td>
<td>65,000</td>
</tr>
</tbody>
</table>

Table 5.7. Service Ceiling of several jet aircraft
5. Maximum Operating Altitude (MOA): Based on Federal Aviation Regulations (FAR) Part 25 [12] Section 1527, the maximum operating altitude is defined as "the altitude for which operation is allowed, as limited by flight, structural, powerplant, functional, or equipment characteristics". For instance, at the maximum operating altitude, the pressurized cabins and compartments must be able to provide a cabin pressure altitude of not more than 8,000 feet (i.e., a 7.8 psi cabin differential pressure). Another requirement is to be able to do an emergency descent to 15,000 ft from max operating altitude in 4 minutes. The maximum operating altitude is more of an equipment limitation than a performance limitation.

Figure 5.27 depicts three types of ceilings (absolute ceiling; service ceiling; and cruise ceiling) with their features. For a military fighter, the absolute ceiling is one the primary performance and design criteria, while cruise ceiling is a primary performance and design criterion for civil transport airplane. In majority of airplanes, service ceiling is about 90% of absolute ceiling, and cruise ceiling is about 80% of absolute ceiling. In general, the ceiling of a jet aircraft is higher than the ceiling of a propeller-driven aircraft.

Table 5.7 [5, 7] presents the service ceiling for several jet aircraft. The maximum operating altitude in the regional jet airliner Embraer ERJ 145 is 37,000 ft; because that's the highest the plane can go while maintaining a 7.8 psi cabin differential pressure, even though the engines would have no problem climbing higher. The maximum operating altitude of an airliner Boeing 747-400 with a normal payload is 41,000, while the service ceiling of the 747-8 is 43,000 feet, and the absolute ceiling is 45,100 ft.

5.6.2. Calculation
In this section, the technique to develop an equation for calculation of absolute ceiling is explained. The following example illustrates how to apply this technique. In any straight line steady level flight, the equilibrium of forces along x-axis yields:

\[ D = T \cos(\alpha + i) \]  \hspace{1cm} (5.7)

For simplicity, the influence of the aircraft angle of attack (\( \alpha \)) and engine setting angle (\( i \)) is neglected, so the engine thrust is assumed to be equal to drag:

\[ T = D \]  \hspace{1cm} (2.46)

As presented in Chapter 4, the available engine thrust for a jet engine is a function of air density. Here, we treat only turbofan engines; the interested reader may use this technique to develop a similar expression for turbojet engine. The variation of turbofan engine thrust with altitude is approximated by the following empirical equations (Equation 5.24). For the troposphere:

\[ T_{\max} = T_{\max{SL}} \left( \frac{\rho}{\rho_o} \right)^{1.2} \]  \hspace{1cm} (Troposphere)

\[ T_{\max} = T_{\max{SL}} \left( \frac{\rho}{\rho_{11000}} \right)^{1.2} \left( \frac{\rho_{11000}}{\rho_o} \right)^{0.2} \]  \hspace{1cm} (Stratosphere)

or

\[ T_{\max} = T_{\max{SL}} \left( \frac{\rho_{11000}}{\rho_o} \right)^{0.2} \left( \frac{\rho_{11000}}{\rho_o} \right)^{1.2} \]  \hspace{1cm} (Stratosphere)

where \( T_{\max} \) denotes the sea-level maximum thrust, and \( T_{\max{SL}} \) is the maximum thrust at altitude, and \( \rho_{11000} \) is the air density at 11,000 m altitude. The drag force is also a function of air density:
\[ D = \frac{1}{2} \rho V^2 SC_D \]  

(2.4)

With respect to figure 5.26, if an aircraft is planned to fly up to its ceiling, it must use its maximum thrust. Furthermore, it must produce a minimum drag. In a technical term, the prerequisite for the absolute ceiling is that, the maximum engine thrust is equal to the aircraft minimum drag:

\[ T_{\text{max}} = D_{\text{min}} \]  

(5.146)

As discussed in section 5.3.3, an aircraft will have its minimum drag, if it flies with the minimum drag speed:

\[ D_{\text{min}} = \frac{1}{2} \rho V_{\text{min}}^2 SC_{D_{\text{min}}} = \frac{1}{2} \rho_o V_{\text{min}}^2 SC_{D_{\text{min},o}} \]  

(5.147)

Subscript E denotes Equivalent airspeed and subscript T denotes True airspeed. We previously derived an equation for minimum drag true airspeed (Equation 5.32). The equivalent minimum drag airspeed is:

\[ V_{\text{min},E} = \sqrt{\frac{2 m_{ac} g}{\rho_o S \sqrt{C_{D_E}}}} \]  

(5.148)

where \( m_{ac} \) is the maximum aircraft mass at the arrival to the absolute ceiling, and \( C_{D_{\text{min},o}} = 2 C_{D_o} \)

Now we substitute equations 5.144, 5.36, and 5.24 into equation 5.143. Since air density (\( \rho \)) in this equation is at the absolute ceiling (ac), the new parameter \( \rho_{ac} \) is used instead.

\[ T_{\text{max},ac} \left( \frac{\rho_{ac}}{\rho_o} \right)^{1.2} = \frac{1}{2} \rho_o V_{\text{min}}^2 D_e S \left( 2 C_{D_o} \right) \]  

(First layer)  

(5.149)

\[ T_{\text{max},ac} \rho_{ac} \left( \frac{\rho_{11000}}{\rho_o} \right)^{0.2} = \frac{1}{2} \rho_o V_{\text{min}}^2 D_e S \left( 2 C_{D_o} \right) \]  

(Second layer)  

(5.150)

where \( \rho_{ac1} \) and \( \rho_{ac2} \) represent the air density at the absolute ceiling for the first and second layer of the atmosphere (troposphere and stratosphere) respectively. In these pair of equations, the only unknown is the air density at the absolute ceiling, \( \rho_{ac} \). Solving these equations for air density at the absolute ceiling (\( \rho_{ac} \)), we will obtain a closed-form solution

\[ \rho_{ac1} = \left( C_{D_o} \left( \frac{\rho_o}{\rho_{ac1}} \right)^{2.2} V_{\text{min}}^2 D_e S \right)^{1/12} \]  

(First layer)  

(5.151)

\[ \rho_{ac2} = \frac{C_{D_o} \left( \rho_o \right)^{2.2} V_{\text{min}}^2 D_e S}{T_{\text{max},ac} \left( \rho_{11000} \right)^{0.2}} \]  

(Second layer)  

(5.152)

When the minimum drag velocity is greater than the stall speed, we may further simplify the equations. Plugging the minimum drag equivalent velocity (equation 5.148) into equations 5.151 and 5.152:

\[ \rho_{ac1} = \left( C_{D_o} \left( \frac{\rho_o}{\rho_{ac1}} \right)^{2.2} \frac{2 m_{ac} g}{\rho_o S \sqrt{C_{D_o}}} \right)^{1/12} \]  

(First layer)  

(5.151)
\[ C_{D_o}(\rho_o)^2 \frac{2m_{ac}g}{\rho_o S} S \]

\[ \rho_{ac2} = \frac{T_{\max_{st}}(\rho_{11000})^{0.2}}{K (\rho_{11000})^{0.2}} \]  
(Second layer) (5.152)

which simplify to:

\[ \rho_{ac1} = \rho_o \left( \frac{2m_{ac}g \sqrt{K C_{D_o}}}{T_{\max_{st}}} \right)^{1/2} \]  
(First layer) (5.153)

\[ \rho_{ac2} = \frac{2m_{ac}g \sqrt{K C_{D_o}} (\rho_o)^{1/2}}{T_{\max_{st}}(\rho_{11000})^{0.2}} \]  
(Second layer) (5.154)

However, the term $2\sqrt{K C_{D_o}}$ is used the calculation of the maximum lift-to-drag ratio (Equation 5.24). Plugging the equivalent of $2\sqrt{K C_{D_o}}$ into equations 5.153 and 5.154 and rearranging the terms yields other forms for equations of absolute ceiling.

\[ \rho_{ac1} = \rho_o \left( \frac{W_{ac}}{T_{\max_{st}}(L/D)_{max}} \right)^{1/2} \]  
(First layer) (5.155)

\[ \rho_{ac2} = \frac{W_{ac}(\rho_o)^{1/2}}{T_{\max_{st}}(\rho_{11000})^{0.2}(L/D)_{max}} \]  
(Second layer) (5.156)

where $W_{ac}$ is the maximum aircraft mass at the arrival to the absolute ceiling. Recall that the maximum lift-to-drag ratio is:

\[ \left( \frac{C_{L}}{C_{D}} \right)_{\max} = \frac{1}{2\sqrt{K C_{D_o}}} \]  
(5.24)

The $m_{ac}$; the maximum aircraft mass/weight at the arrival to the absolute ceiling is determined by deducting the fuel consumed during take-off and climb from take-off mass/weight:

\[ m_{ac} = m_{TO} - t_{cl}C \frac{T_{cl}}{g} - m_{fro} \]  
(5.157)

\[ W_{ac} = m_{ac}g \]  
(5.158)

where $t_{cl}$ is the time to climb to the absolute ceiling, $C$ is the specific fuel consumption (in 1 per unit time), and $m_{fro}$ is the mass of fuel burnt during take-off phase. The parameter $T_{cl}$ denotes the average engine thrust during the climb phase.

The equations 5.153 and 5.155 are employed to determine absolute ceiling if located at the first layer (Troposphere), while the equation or 5.154 and 5.156 governs the absolute ceiling at the second layer (Stratosphere). Since a non-air-breathing aircraft can fly higher than the second layer, there is no need to develop the absolute ceiling equation for beyond the second layer.

In equation 5.155 and 5.156, \( \rho_{ac1} \) and \( \rho_{ac2} \) denote air density at the absolute ceiling for first and second layer respectively, \( \rho_{11000} \) denote air density at altitude of 11,000 ft, and \( T_{\max_{st}} \) is the maximum available engine thrust at sea level. Note that the subscript E denotes equivalent airspeed. When the air density is determined from one of these two equations, we will refer to atmospheric tables (such as appendix A or B) to find its corresponding altitude; that represent the absolute ceiling.
In practice, it is a wise practice to first assume that the absolute ceiling is located within the first layer. In such case, use the equation 5.153 or 5.155. If the calculation yields an answer ($\rho_{ac1}$, air density) that corresponds to the second layer, the results is not valid. Thus, you will have to repeat the calculation by using the equation 5.154 or 5.156. The new result ($\rho_{ac2}$, air density) would be the right answer.

Please note that, if the minimum drag equivalent airspeed is less than the sea level stall speed, you cannot use equations 5.153 and 5.155, 5.154 and 5.156. In such case, assume a value for the minimum drag equivalent airspeed greater than the sea level stall speed:

$$V_{\text{min}_{DG}} = kV_s$$  \hspace{1cm} (5.159)

where $1.1 < k < 1.2$. Then, use equation 5.151 or 5.152.

The equations 5.153 and 5.154 imply that the absolute ceiling is an inverse function of aircraft mass, zero lift drag coefficient, and induced drag factor. Moreover, the absolute ceiling is a direct function of engine thrust. In order to increase the absolute ceiling, one must increase a better engine with a higher thrust. In addition, the aircraft mass should be decreased by making the aircraft lighter. Moreover, the aircraft should be more aerodynamic by reducing the zero lift drag coefficient. Since the induced drag factor is a function of aspect ratio, we also may conclude that a higher aspect ratio will lead to a higher absolute ceiling.

We may conduct another interesting conclusion, regarding the relationships between weather, and location, with the absolute ceiling. A jet aircraft will have a higher absolute ceiling during a warmer weather (say summer) compared with a colder time (say winter). In addition, a jet aircraft has a higher absolute ceiling when flying over a colder region (i.e., closer to poles), compared to flying over a warmer area (i.e., close to equator). For instance, a jet aircraft will have a higher absolute ceiling when flying over Mexico compared with flying over Alaska.

The equations 5.153 and 5.154 illustrate that the absolute ceiling is not a function of wing area, and wing loading. This implies that a small aircraft (i.e., small wing area) may have a higher absolute ceiling than a large aircraft; provided it has a sufficient engine thrust. Since the absolute ceiling is a function aircraft mass; as the time elapses, the absolute ceiling is gradually increased. This is due to the fact that the fuel is gradually consumed and the aircraft weight is gradually dropped.

Equations 5.155 and 5.156 indicate the direction to an aircraft designer the ways to improve the absolute ceiling. In order to increase the absolute ceiling, one must apply a change such that the density will decrease (i.e., height is increased). First of all, since the $T_{\text{max}}$ appears in the denominator, one must employe an engine with a higher thrust. Second alternative is to reduce the $C_{D_{0}}$ by making the aircraft more aerodynamic. Third technique is to reduce the aircraft mass. The last technique to improve the absolute ceiling is to reduce the induced drag factor ($K$); which implies an increase in the wing aspect ratio.

**Example 5.9**
Consider a business jet aircraft equipped with a turbofan engine, a mass of 11,000 kg 35 m$^2$, a wing area, and a sea level engine thrust of 33 kN. Assume the climb to the absolute ceiling takes 25 minutes, and the average engine thrust during climb is equal to 90% of the maximum thrust at sea level. Other aircraft specifications are:

$K = 0.055$, $C_{D_{0}} = 0.018$, $C_{L_{\text{max}}} = 2.2$, $C = 0.8$ N/hr/N.
Determine the absolute ceiling for this aircraft in an ISA condition. Ignore the effect of the fuel consumed during take-off.

Solution:

$$m_{ac} = m_{TO} - t_c C \frac{T_{el}}{g} - m_{fro} = 11000 - 0.2 \times 0.8 \times \frac{0.9 \times 33000}{9.8} - 0 = 101,92.4 \text{ kg}$$  (5.157)

The minimum drag equivalent airspeed:

$$V_{\text{min}_{ac}} = \sqrt{\frac{2mg}{\rho_o S C_{D_{\text{c}}}} \sqrt{\frac{C_{D_{\text{c}}}}{K}}} = \frac{2 \times 10192.4 \times 9.81}{1.225 \times 35 \times 0.018} \Rightarrow V_{\text{min}_{ac}} = 90.23 \text{ m/sec}$$  (5.32)

The stall speed at sea level is:

$$V_s = \sqrt{\frac{2mg}{\rho_o S C_{L,\text{max}}}} = \frac{2 \times 10192.4 \times 9.81}{1.225 \times 35 \times 2.2} \Rightarrow V_s = 89.5 \text{ m/sec}$$  (2.71)

The minimum drag speed is greater than the sea level stall speed, so we can use equation 5.153 or 5.154. It is initially assumed that the absolute ceiling is located within the first layer:

$$\rho_{ac1} = \rho_o \left( \frac{2mg}{T_{\text{max}_{ac}}} \right)^{1/2} = 1.225 \left( \frac{2 \times 10192.4 \times 9.81 \times \sqrt{0.055 \times 0.018}}{33000} \right)^{1/3} = 0.3078 \text{ kg/m}^3$$  (5.153)

The corresponding altitude to this density ratio from appendix B is obtained as 12,090 m. This altitude lies within the second layer, but our assumption was to have absolute ceiling at the first layer. Therefore, the result is not valid, and we have to repeat the calculation by using the equation that yields an altitude within the second layer. From Appendix A, the air density at 11,000 m is 0.3638 kg/m³.

$$\rho_{ac2} = \frac{2mg}{T_{\text{max}_{ac}} \left( \rho_{1000} \right)^{1/2}} = \left( \frac{2 \times 10192.4 \times 9.81 \times \sqrt{0.055 \times 0.018 \times 1.225^{1/2}}}{33000 \times 0.3638^{0.2}} \right) = 0.2977 \text{ kg/m}^3$$  (5.154)

The altitude corresponding to this air density is 12,290 m or 40,321 ft. Therefore, the absolute ceiling of this jet aircraft is 40,321 ft.

### 5.7. Cruise Performance

As shown in figure 5.1, the major phase of the mission of a transport aircraft is cruise, which is also the longest phase. For a transport aircraft, the cruise phase of flight consumes the majority of fuel. Due to this reason, transport aircraft are usually designed for an optimum performance at their cruise speed; which frequently means the longest range. The cost of flight for a transport aircraft can be minimized by optimizing the cruising flight. A number of wing parameters such as airfoil section, setting angle are determined primarily based on the cruise performance [10]. Compared with other phases of flight, the cruise is historically the safest phase of a flight. In this section, the analysis of three topics: 1. cruise altitude, and 2. cruise speed 3. cruise ceiling will be addressed.

#### 5.7.1. Cruise Speed

In section 5.3.1, some explanation about the significance and calculation of the aircraft maximum speed was presented. In a long duration flight, for the economic reasons and maintenance considerations, the aircraft are not usually flying with the maximum speed. They
are flying with an efficient speed called cruising speed. This speed is lower than (around 70%-90% of) the maximum speed. In this flight condition, full throttle is not utilized. The other reason is to let the engine lasts longer. It is also recommended to car drivers not to use maximum throttle in a long trip; and not to drive with highest speed for a long time. The reason is that it may hurt the engine due to high temperature and high fuel consumption. The reason for flying with cruise speed is very similar to that of the car driving.

The amount of usage of jet engine thrust in a cruising flight depends on parameters such as aircraft weight, flight altitude, and aircraft mission. This could be between 60% up to about 90% of the maximum thrust. Thus, the cruising speed is always lower than maximum speed. Table 5.3 demonstrates cruise speed of several jet aircraft.

Typical cruising airspeed for long-range commercial passenger flights is 440–500 knot (Mach 0.75–0.85). The cruising speed of a large transport aircraft such as Boeing 747 and Airbus 340 are about Mach 0.85. In transonic speed, particularly close to Mach 1, shock waves are strong and they produce wave drag. Therefore, it is undesirable to fly with a cruising speed that is very close to Mach 1. The cruising speed for the Boeing 747-400 is Mach 0.8, while for the Boeing 787 Dreamliner is Mach 0.85, and for the Boeing 767 is Mach 0.84.

The business tri-jet aircraft Dassault Falcon 50 with a maximum take-off mass of 18,000 kg has maximum operating speed of 350 knot at sea level, while 370 knot at 23700 ft. it also has a maximum cruising speed of 487 knot (Mach 0.85), a normal cruising speed of 459 knot (Mach 0.8), and a long range cruising speed of 430 knot (Mach 0.75) at 35,000 ft.

![Figure 5.28. Cursing speed with various thrust lines](image)

5.7.1.1. Based on Engine Chart

Figure 5.28 demonstrates the variation of drag force versus speed as a parabolic curve. In this figure, several thrust lines are also presented. Please note that for simplicity, it is assumed that the engine thrust is constant with respect to airspeed. The intersections between each thrust line
and drag curve demonstrate two possible solutions. The solution with a higher value (highest speed) is the cruising speed.

Based on the derivation provided in the previous section, and by considering the equation 5.23, we are able to develop a relationship to determine the cruising speed of an aircraft as follows:

$$AV_c^2 + \frac{B}{V_c^2} - nCT_{max} = 0$$

(5.160)

where the coefficients A, B, and C are found from equations 5.24 through 5.26. The parameter “n” is the percentage of usage of engine thrust and is in the following range:

$$0.9 > n > 0.65$$

(5.161)

The coefficient "n" is a factor of several parameters such as fuel cost, engine type, engine maintenance, flight duration, market necessity, flight mission, aircraft weight, and altitude. It is determined through an optimization process to optimize few parameters at the same time (i.e. minimize fuel cost, minimize flight duration, maximize engine life, minimize engine maintenance cost, and compete in the market). This parameter as a function of altitude and aircraft weight is usually provided to pilots by the aircraft manufacturer (in pilot operating handbook).

![Figure 5.29. Cruising altitude of Boeing 747 (with JT9D 7A engine) at various weights](image)

**Example 5.10**

The following jet aircraft is equipped with a turbojet engine and is utilizing 80% of its maximum engine thrust in a cruising flight at 30,000 ft.

$$m = 47,000 \text{ kg}, \quad C_{Do} = 0.022, \quad T_{maxSL} = 103.16 \text{ kN}, \quad S = 127 \text{ m}^2, \quad K = 0.047,$$

Determine the cruising speed of this aircraft at 30,000 ft altitude.
Solution:
From atmospheric table in appendix B, the air density at the altitude of 30,000 ft is:
\[ \rho = 0.347 \times 1.225 = 0.458 \frac{kg}{m^3} \]
The cruising speed can be obtained by using equation 5.28:
\[
AV_c^2 + \frac{B}{V_c^2} - 0.8CT_{max} = 0
\]  
(5.160)

The maximum turbojet engine thrust at 30,000 ft is:
\[
T = T_{max,SL} \left( \frac{\rho}{\rho_0} \right)^{0.9} = 103160 \times (0.374)^{0.9} = 42569.2 \ N
\]  
(4.21)

The coefficients A, B, and C are (n is 0.8):
\[
A = \frac{1}{2} \rho SC_{D_x} = 0.5 \times 0.458 \times 127 \times 0.022 = 0.64
\]  
(5.29)
\[
B = \frac{2KW^2}{\rho S} = \frac{2 \times 0.047 \times (47000 \times 9.81)^2}{0.458 \times 127} = 343551924
\]  
(5.30)
\[
C = \left( \frac{\rho}{\rho_0} \right)^{0.9} = (0.374)^{0.9} = 0.3857
\]  
(5.32)

Substitution yields:
\[
0.64V_c^2 + \frac{343551924}{V_c^2} - 0.8 \times 0.3857 \times 103160 = 0
\]  
(5.160)

Solution of this algebraic equation yields the following acceptable result:
\[
V_c = 184.13 \ m/sec = 357 \ KTAS
\]

5.7.1.2. Based on Range Mission
In case, where, the engine performance chart for the optimum performance (best range) is not provided, the theoretical value of the cruise speed is determined by using the lift coefficient for the maximum range, and the cruising altitude. In a constant altitude cruise, the lift is equal to weight; particularly when the aircraft is cruising for the purpose of maximizing range:
\[
W = L_{max_x} = \frac{1}{2} \rho C V_c^2 SC_{L_{max}}
\]  
(5.162)

When the cruise altitude is known, we can obtain the cruise velocity, based on the velocity to maximize range (see equation 5.90) as follows:
\[
V_c = \sqrt{\frac{2W}{\rho C SC_{L_{max}}}}
\]  
(5.163)

where the cruise lift coefficient (\(C_{L_c}\)) is determined by equation 5.91:
\[
C_{L_c} = C_{L_{max}} = \sqrt{\frac{C_{D_v}}{3K}}
\]  
(5.164)

Typical values of the cruise lift coefficient for majority of civil subsonic aircraft vary from 0.1 to 0.5. These cruise lift coefficients correspond to the aircraft angle of attacks of about 1 to 5 degrees. Most large transport aircraft (e.g., Boeing 747 and Airbus 380) have about 4-5 degrees of angle of attack at the beginning of cruise, while about 2-3 degrees of angle of attack at the end of cruising flight. Airbus A-320 with a maximum take-off mass of 77,000 kg has a
maximum operating speed of Mach 0.82 (350 knot at ISA condition) while the optimum cruising speed is Mach 0.78.

Equation 5.163 implies that the cruise speed increases with altitude (via $\rho_c$), and decreases with aircraft weight. In addition, the value of aircraft cruise speed is dependent on inverse of wing area (indeed, its square root): as the wing area is decreased, the cruise velocity is increased. Furthermore, the aircraft cruise speed is a direct function of square root of the wing loading (W/S). The following section provides the technique to determine cruise altitude.

5.7.2. Cruise Altitude

Another important parameter affecting the cruising performance is the cruise altitude which is the level portion of aircraft travel where the flight is most fuel efficient. The cost of flight at various altitudes depends on the speed. To optimize the cruise performance, a jet aircraft must be flown with a specific cruising speed at each altitude. The cruising altitudes of different aircraft are not the same. This altitude is a function of several parameters such as aircraft weight, aviation regulations and the distance to the destination. In 2014, there are 87,000 daily aircraft flights [14] in the United States. This large number produces great limits and challenges for flight engineers, airliners to determine the safe while economic cruise altitudes for various aircraft.

In fact, other than practical factors (such as traffic, and safety regulations), the cost of flight is the main driver to determine the cruise altitude. The calculation of the cruise altitude is a challenging problem, and should take into account a number of parameters simultaneously. In this section, the procedure to determine the cruise altitude is reviewed, and related important charts and figures are discussed.

Since in higher altitude (above 18,000 ft), the air pressure is not enough for human to have a normal breath, the compressed air must be provided for pilot, crew and passengers. Normally at altitude higher than 12,000 ft, the aircraft needs to be provided with an air conditioning system. In busy air traffic, because of the safety reasons, all aircraft cannot fly at the same altitude. Thus, aviation regulations determine the cruising altitude of each aircraft (altitude separation requirements).

One of the FAA regulations is the “vertical separation” which states that that all Eastbound flying aircraft must use altitude with odd numbers of 1,000 ft, while all returning aircraft (Westbound) are required to fly at altitudes that are even numbers of 1,000 ft. This leads to a safety measure of 1,000 ft altitude difference between two aircraft with opposite directions. The standard rule defines an East/West track split: 1. Eastbound – Magnetic Track 000 to 179° – odd thousands (FL 250, 270, etc.), 2. Westbound – Magnetic Track 180 to 359° – even thousands (FL 260, 280, etc.).

Between the surface and an altitude of 29,000 feet (8,800 m), no aircraft should come closer vertically than 300 meters, unless some form of horizontal separation is provided. Above 29,000 feet (8,800 m), no aircraft shall come closer than 600 m (or 2,000 feet), except in airspace where “reduced vertical separation minimum (RVSM)” can be applied. Above FL 410- 2,000 feet, except: 1. In oceanic airspace, above FL 450 between a supersonic and any other aircraft- 4,000 feet. 2. Above FL 600 between military aircraft- 5,000 feet.

The cruising altitude of most General Aviation (GA) aircraft is about 10,000 ft to 20,000 ft altitude. The cruising altitude of most transport aircraft are about 20,000 ft to 40,000 ft altitude.
The cruising altitude of most fighter aircraft is about above 40,000 ft to 60,000 ft altitude. Figure 5.29 shows the cruising altitude of transport aircraft Boeing 747 at various aircraft weights.

The airliner Airbus A-320 with a maximum take-off mass of 77,000 kg has an initial cruise altitude of 37,000 ft, while the maximum certified altitude is 39,800 ft. The business tri-jet aircraft Dassault Falcon 50 with a maximum take-off mass of 18,000 kg has an initial cruise altitude of 41,000 ft, while the maximum certified altitude is 49,800 ft. The business jet Embraer Phenom 300 has a maximum operating altitude of 45,000 ft with a cruising speed of 450 knot.

As figures 5.18 and 5.19 illustrates, higher altitudes are more efficient for additional fuel economy. In general; for operational and air traffic control reasons; FAA requires each aircraft to fly at a constant altitude throughout its cruising flight. On long range flights, the pilot may climb from one flight level to a higher one as permission is requested and given from air traffic control authorities. This operation is called a step climb. There is an optimum cruise altitude for a particular aircraft type and each flight condition including payload weight, air temperature, and flight distance.

Typical cruise altitude for high subsonic large transport aircraft is about 30,000-40,000 ft. The B-747-400 has an initial cruising altitude of 35,000-38,000, when taking off with the maximum take-off weight, while the Boeing 777 (2GB) has an initial cruising altitude of 39,400 ft. The maximum certified altitude for Airbus 320 [7] is 39,800 ft.

As we classified earlier, there are three following options in a cruising flight: 1. Constant altitude-constant lift coefficient flight; 2. Constant airspeed-constant lift coefficient flight; and 3. Constant altitude-constant airspeed flight. These can be regrouped into two groups: 1. Constant altitude flight (case 1 and 3); 2. Non constant-altitude flight (case 2). For the case 1 and 3, the technique to determine the cruising altitude is presented, while for a cruise-climb, the method to calculate the initial cruise altitude will be discussed.

![Diagram](image)

**Figure 5.30. Variations of flight cost with respect to altitude**

Although cruise-climb flight can considerably increase the range of an aircraft for long-range flights, it does involve a continuous increase of altitude that is not consistent with safe flight when the presence of other aircraft must be considered. Consequently, the opportunity to use
cruise-climb flight is limited by FAA regulations. However, on long-range flights, stepped-altitude flight may be employed, which is a series of constant altitude-constant airspeed flight segments conducted at different altitudes (with each step to be about 2,000 ft). Stepped-altitude flight is often used on long-range flights, such as transcontinental and transoceanic flights. This program will place a burden on air traffic control authorities to ensure a safe clearance from any altitude crossed during climb to new cruise altitude.

As we derived in Section 5.6, at absolute ceiling, the lift-to-drag ratio is equal to the aircraft maximum lift-to-drag ratio. Equation 5.93 implies that, the lift-to-drag ratio for a constant speed flight to maximize range is only 15.5% greater than the maximum lift-to-drag ratio, as is required for level flight, but not much more.

\[
\left( \frac{L}{D} \right)_{\text{max}} = 1.155 \left( \frac{L}{D} \right)_{R\text{max}}
\]  

(5.165)

This indicates that the altitude for maximum range flight lies below; but close to; the absolute ceiling for the associated throttle setting. As equation 5.8 demonstrates, at cruise altitude, the lift is almost equal to weight:

\[
W = L + T \sin(\alpha + i_c)
\]  

(5.166)

If the contribution of the engine thrust due to a low angle of attack and low setting angle is ignored, we can write:

\[
\frac{T_{\text{max}}}{W} = \frac{1}{(L/D)_{\text{max}}} = \frac{1}{0.866(L/D)_{\text{max}}} = \frac{1.155}{(L/D)_{\text{max}}}
\]  

(5.167)

Equation 5.157 implies that, the thrust-to-weight ratio for a constant speed flight to maximize range is only 15.5% greater than the reciprocal of the maximum lift-to-drag ratio, but not much more. So, the required thrust at cruise altitude will be:

\[
T_{\text{max}} = 1.155\frac{W}{(L/D)_{\text{max}}}
\]  

(5.168)

In Chapter, the variation of turbofan engine thrust with altitude is approximated by

\[
T = T_0 \left( \frac{\rho}{\rho_0} \right)^{1.2}
\]  

(4.24)

At cruise altitude, the thrust in equations 5.168, and 4.24 are the same. Equating these two equations yields:

\[
\frac{1.155W}{(L/D)_{\text{max}}} = T_0 \left( \frac{\rho}{\rho_0} \right)^{1.2}
\]  

(5.169)

The only unknown variable in this equation is the cruise altitude air density, which is obtained as:

\[
\rho_c = \left( \frac{1.155W\rho_0^{1.2}}{(L/D)_{\text{max}} T_{SL}} \right)^{\frac{1}{1.2}}
\]  

(5.170)

For an aircraft with the turbojet engine, the reader is encouraged to develop a similar expression. Using appendices A or B, one can determine the corresponding cruise altitude (h_c).

Equation 5.170 implies that the cruise altitude (via \( \rho_c \)) depends on aircraft weight, engine thrust, and maximum lift-to-drag ratio. As the engine thrust and maximum lift-to-drag ratio increase, the cruise altitude would increase. Since the maximum lift-to-drag ratio is an inverse function of aircraft zero-lift drag coefficient (C_{Do}), the more aerodynamic aircraft result in a
higher cruise altitude. This reality drove the design of high altitude reconnaissance aircraft such as U-2 and Lockheed SR-71 Blackbird (Figure 5.13). Their jet engines should have been powerful enough to support +60,000 ft cruise altitude. Furthermore, the zero-lift drag coefficient (C\textsubscript{D\textsubscript{0}}) should be low enough to allow the aircraft to fly at very high altitude.

However, as the aircraft weight increases, the cruise altitude will drop; due to a decrease in the air density (\(\rho\)). This indicates the importance of reducing aircraft weight during design phase. We also note that the cruise altitude is direct function of the thrust-to-weight ratio (T/W). As the thrust-to-weight ratio increase, the cruise altitude increase. This show the need to a high thrust-to-weight ratio for a high altitude flight.

For maximum range flight, the instantaneous fuel consumption in weigh of fuel per unit time is:

\[
\left(\frac{dW_f}{dt}\right)_{\text{Rmax}} = CT_{\text{max}} = \frac{1.155CW}{(L/D)_{\text{max}}} \quad (5.171)
\]

The specific fuel consumption decreases slowly with altitude, reaching its minimum value at the tropopause and then increasing even more slowly in the second layer (stratosphere). Furthermore, the thrust-to-weight ratio is greater than the reciprocal of the maximum lift-to-drag ratio, as is required for level flight, but not much more. This indicates that the altitude for maximum range flight lies below; but close to; the absolute ceiling for the associated throttle setting. Moreover, the absolute ceiling of most large transport aircraft is about 40,000-50,000 ft. Thus, the maximum range altitude is a few thousands foot (approximately 5,000 ft) below the absolute ceiling for that particular throttle setting. Therefore, there is a slight advantage to flying in the vicinity of the tropopause, all other things being equal.

Another advantage of cruising flight in the vicinity of the tropopause is the lack of strong wind and turbulence. A potential drawback to fly at the vicinity of tropopause; is the rare instance of violent sudden drop in air pressure and severe turbulence; which causes an aircraft to lose altitude rapidly. This will cause the passengers without seatbelt to hit the ceiling which results in their injury and hospitalization. For instance; on July 21, 2010; thirty passengers injured on United Airlines flight; when suddenly the plane lurched into a free-fall; due to severe turbulence; sending people literally flying into the cabin ceiling.

For an accurate calculation of the best cruise altitude, the entire mission should be considered [11]. The cruise altitude with lowest cost is theoretically the best altitude (Figure 5.30). When the practical limits and considerations are taken into account, the optimum cruise altitude will be determined. Charts in the aircraft operating handbooks (e.g., [5]) allow the pilot to select the best combination of altitude/speed/range/endurance for the intended flight.

Overall fuel cost (S) of flight is the sum of the cost for five major segments of the flight 1. Take-off and taxi, 2. Climb, 3. Cruise, 4. Descent, 5. Landing:

\[
S = S_{TO} + S_c + S_{cl} + S_d + S_L \quad (5.172)
\]

The cost of fuel for each segment is determined by multiplying amount of fuel burnt per segment by fuel cost per unit weight/mass (f\textsubscript{c}).

\[
S = W_f f_c \quad (5.173)
\]

In 2014, a gallon of jet fuel is estimated to be about $8 which is about $2 per liter. The amount of fuel burnt per segment is calculated by using the definition of specific fuel consumption (Equation 5.55):
where t is the duration of flight for a segment, T denotes the average engine thrust in the segment, and C is the fuel consumption for the segment. The regional jet airliner ERJ-145 with a maximum take-off mass of 20,600 kg [5] has a time to climb to 35,000 ft altitude (FL350) of 20 minutes, while this time for Emraer 175 a maximum take-off mass of 37,500 kg is 16 minutes. The business tri-jet aircraft Dassault Falcon 50 with a maximum take-off mass of 18,600 kg has a time to climb to 41,000 ft altitude (FL410) of 23 minutes. The jet fighter Dassault Mirage 2000 has a time to climb to 11,000 m (36,080 ft) and Mach 1.8 of approximately 5 minutes.

Recall that during cruise, the thrust is almost equal to drag, as we derived earlier (equ 5.7); it is repeated here for convenience:

\[ D = T \cos(\alpha + i) \] (5.7)

As the cruise altitude is increased, the cost for climb is increased, but the cost for cruise is decreased. This is due to the fact that the drag and thrust decrease with altitude. In addition, as the cruise altitude is increased, the optimum value for the parameter “M L/D” is increased (See Figure 5.19).

Due to five segments for a flight of a transport aircraft (Figure 5.1), the total consumed weight of fuel is determined by:

\[ W_f = t_{TO} \cdot T_{TO} \cdot C_{TO} + t_{cl} \cdot T_{cl} \cdot C_{cl} + t_{cr} \cdot T_{cr} \cdot C_{cr} + t_d \cdot T_d \cdot C_d + t_L \cdot T_L \cdot C_L \] (5.175)

Please note that the engine thrust and fuel consumed during descent and landing is minimal; and may be neglected for simplicity. Large transport aircraft usually use reverse thrust to reduce the landing run; thus a minimal amount of fuel is burnt during landing. As an approximation, you may consider that about 5% of the total fuel is been consumed during take-off, descent, and landing; thus the equation 5.175 is simplified to:

\[ W_f = 1.05 \left( t_{cl} \cdot T_{cl} \cdot C_{cl} + t_{cr} \cdot T_{cr} \cdot C_{cr} \right) \] (5.176)

Figure 5.31 illustrates a comparison between two flight operations with different cruise altitudes. Both flights have the similar take-off and landing distances and durations. However, the climb and descent ground distances and durations are different. One reason is that the rate of climb decreases with altitude, so the average climb angle to fly to a higher altitude is less than that for a lower altitude. Similarly, the ground distance to fly to a higher altitude is greater than that for a lower altitude. To determine the optimum cruise altitude, you need to write a computer program to calculate the fuel required to fly for a mission for various cruise altitudes. The altitude that requires the lowest fuel weight for the entire flight is selected as the best cruise altitude.

Various parameters influence the cruise altitude; including flight operating cost, flight time, flight distance, and competition among airliners. The cruise flight time for the constant altitude-constant airspeed flight program can easily be found by dividing the range by the airspeed. The same procedure can also be used for the cruise-climb flight time.

The time required for any of the maximum range in the cruise segment, for constant lift coefficient flight programs (either constant lift coefficient-constant speed; or constant lift coefficient-constant altitude) can be determined by inverting equation 5.62 and integrating over the flight path from \( W_1 \) to \( W_2 \) to obtain:
\[ t_{\text{max},g} = \frac{R_{\text{max}}}{V_{\text{max},g}} = \frac{0.866(L/D)_{\text{max}}}{C} \ln \left( \frac{1}{1 - G} \right) \] (5.177)

Figure 5.31. Comparison between two flight operations with different cruise altitudes

Figure 5.32. Variations of cruise altitude with range and cruise velocity
The time of flight is the same for all maximum range, constant lift coefficient flight programs at any altitude; i.e., it is independent of the actual range flown. This implies that the cruise at a higher altitude (which delivers a longer range) will have the same duration when flown at lower altitude (which delivers a shorter range). This indicates that, with a full fuel tank, the cost for cruise at higher altitude is less than that for a lower altitude. The range of a transport aircraft is about 50 percent greater at 30,000 ft than at sea level. However, the optimum cruise altitude is determined when the cost for climb is included in the calculation process.

Consider the equation for the maximum range speed (equation 5.x). We note that $V_{\text{max}}$ is inversely proportional to the square root of the air density and thus increases with altitude. We also notice that the thrust required is independent of the altitude itself, being dependent only upon the instantaneous weight of the aircraft and upon maximum lift-to-drag ratio. In other words, the thrust required at sea level is identical (as is the drag) to that required at high altitudes. We fly at altitude to obtain the range benefits (and the secondary benefits of flying above the weather and surface turbulence) accruing to the increase in the airspeed necessary (with the reduced air density) to generate the lift required to counteract the weight and keep the airplane in the air. As we increase the altitude, however, the available thrust does decrease and we must be sure that there is sufficient available thrust at the cruise altitude to satisfy equation 5.68.

The cruise altitude for a cruise-climb flight option is gradually increasing. The altitude at the end of cruise-climb flight ($h_2$) can be expressed in terms of both the initial altitude ($h_1$) and the fuel fraction. The relation between fuel weight, initial weight, and final weight is:

$$ W_2 = W_1 \left( 1 - \frac{W_f}{W_i} \right) \quad (5.178) $$

At both initial cruise altitude, and final cruise altitude, the lift is equal to weight.

$$ W_1 = L_1 = \frac{1}{2} \rho_1 V^2 S C_L \quad (5.179) $$

$$ W_2 = L_2 = \frac{1}{2} \rho_2 V^2 S C_L \quad (5.180) $$

Recall, the airspeed, and lift coefficient are kept constant throughout cruise-climb operation. Plugging equation 5.73 and 5.74 into equation 5.63 yields:

$$ \frac{1}{2} \rho_2 V^2 S C_L = \frac{1}{2} \rho_1 V^2 S C_L \left( 1 - \frac{W_f}{W_i} \right) \quad (5.181) $$

or

$$ \rho_2 = \rho_1 \left( 1 - \frac{W_f}{W_i} \right) \quad (5.182) $$

The density ratio at the end of cruise-climb flight ($\sigma_2$) can be expressed in terms of both the initial density ratio ($\sigma = \rho/\rho_o$) and the fuel fraction, i.e.,

$$ \sigma_2 = \sigma_1 \left( 1 - G \right) \quad (5.183) $$

where, $\sigma_1$ is air density ratio at the beginning of the cruise.

The climb angle in cruise-climb is so small that can be ignored. However, the altitude difference could be considerable (e.g., 10,000 ft). For instance, the aircraft in example 5.6 with the range of 45,050 km in cruise-climb flight program has an 8,500 ft increase in altitude and a climb angle of 0.033 degrees. When the air density at the end of cruise-climb flight ($\sigma_2$)
obtained, one can utilize Appendix A or B to determine final altitude. It should be mentioned that the error in range from using the level constant-altitude flight range equation instead of cruise-climb equation is of order of 1-2 percent.

**Example 5.11**

A jet transport aircraft with a mass of 70,000 kg, a wing area of 125 m², is equipped with two turbofan engines; and has the following characteristics:

\[ C_{D_0} = 0.02, \quad K = 0.05, \quad T_{SL} = 2\times87 \text{ kN} \]

Determine

a. cruise altitude
b. cruising speed in terms of Mach number

Ignore other phases (including the climb phase) of the flight, and assume that this cruise altitude is to maximize the range.

**Solution:**

The maximum lift-to-drag ratio:

\[
\left( \frac{C_L}{C_D} \right)_{\text{max}} = \frac{1}{2\sqrt{KC_{D_e}}} = \frac{1}{2\sqrt{0.05\times0.02}} \Rightarrow \left( \frac{C_L}{C_D} \right)_{\text{max}} = 15.81 \quad (5.24)
\]

The cruise altitude air density:

\[
\rho_c = \left( \frac{1.155W\rho_o^{1.2}}{(L/D)_{\text{max}} T_{SL}} \right)^{\frac{1}{2}} = \left( \frac{1.155(70000\times9.81)1.225^{1.2}}{15.81\times2\times87000} \right)^{\frac{1}{2}} = 0.434 \text{ kg/m}^3 \quad (5.170)
\]

Using appendix A, this air density corresponds to a cruise altitude of 9,600 m or 31,500 ft. The cruise lift coefficient is determined by equation 5.91:

\[
C_{L_{\text{max}},c} = \sqrt{\frac{C_{D_e}}{3K}} = \sqrt{\frac{0.02}{3\times0.05}} = 0.365 \quad (5.91)
\]

Then, the cruise velocity is obtained as follows:

\[
V_c = \sqrt{\frac{2W}{\rho_c SC_{L_{\text{max}},c}}} = \sqrt{\frac{2\times70000\times9.81}{0.434\times125\times0.365}} = 263.15 \text{ m/s} \quad (5.163)
\]

The speed of sound at this altitude is 301.2 m/s, so the cruise Mach number is:

\[
M_c = \frac{263.15}{301.2} = 0.874 \quad (1.25)
\]

**5.7.3. Cruise Ceiling**

Under development
Problems
1. Determine the maximum lift-to-drag ratio of a glider with the following features:
   \[ AR = 25, \ e = 0.9, \ C_{D_0} = 0.012 \]

2. A 20,000 kg jet aircraft is flying at sea level with 4 degrees of angle of attack. Determine how much lift and drag this aircraft has produced at this flight condition, if its engine thrust is 50 kN. Assume the thrust line coincides with fuselage center line.

3. Consider a jet aircraft with the following features:
   \[ T_{\text{max}} = 27 \text{ kN}, \ S = 56 \text{ m}^2, \ m_{\text{TO}} = 11,000 \text{ kg}, \ C_{l_{\text{max}}} = 1.7, \ K = 0.05, \ C_{D_0} = 0.018 \]
   a. Determine the following parameters:
   b. Stall speed
   c. Minimum drag speed
   d. Maximum range speed
   e. Maximum endurance speed
   f. Cruising speed at 20,000 ft altitude with 75% engine thrust
   g. Maximum speed at sea level.

4. Consider a jet transport aircraft with the following features:
   \[ T_{\text{max}} = 757 \text{ kN}, \ S = 476 \text{ m}^2, \ m_{\text{TO}} = 270,000 \text{ kg}, \ C_{l_{\text{max}}} = 2.3, \ K = 0.04, \ C_{D_0} = 0.017, \ m_f = 50,000 \text{ kg}, \ C = 0.7 \text{ lb/hr/lb} \]
   Determine the maximum range when flies at 25,000 ft altitude.

5. Plot variation of maximum speed of aircraft in problem 3 versus altitude up to 40,000 ft. At What altitude the maximum speed is the highest (\(V_{\text{max}}\))?


7. A jet fighter aircraft has the following data:
   \[ T_{\text{maxSL}} = 300 \text{ kN}, \ S = 217 \text{ m}^2, \ m_{\text{TO}} = 63,000 \text{ kg}, \ C_{l_{\text{max}}} = 2.1, \ K = 0.043, \ C_{D_0} = 0.021, \]
   Is this fighter able to fly with Mach 1.7 at 20,000 ft altitude?

8. Determine whether or not the fighter in problem 7 can survive from a missile that can reach 40,000 ft altitude.

9. What would be the speed, if the fighter in problem 7 is required to fly at 25,000 ft with 70% of its engine thrust?

10. Assume that the aircraft in problem 4 has two engines. Is it possible for this aircraft to fly at 24,000 ft, if one engine is inoperative?

11. If the aircraft in problem 4 is required to fly with cruise-climb flight program, what would be its final altitude, when the fuel tanks get empty? Ignore the safety of this flight.
12. Determine the range of the aircraft in problem 4, if carries only half of its maximum fuel and is required to fly with constant-altitude, constant speed.


14. Consider the fighter in problem 7 has fuel capacity of 20,000 kg and specific fuel consumption of 0.8 lb/hr/lb. What would be the range of this fighter, it refuels three times in the air from a tanker.

15. What would be the maximum range, if the fighter in problem 7 carries an external tank with the capacity of 30,000 kg fuel?

16. Consider the aircraft in problem 4 that has consumed 70% of its fuel and is ready to land on a runway. The pilot suddenly receives a massage from airport control tower that the runway is not ready for landing. The pilot finds out that the nearest alternate airport is 600 km away from the current position. Will this aircraft safely land on its new destination?

17. An anti-submarine jet aircraft above Atlantic Ocean is searching for enemy submarine. This aircraft has the following data:

   \[ T_{\text{max}} = 64 \text{ kN}, \quad S = 41 \text{ m}^2, \quad m_{\text{TO}} = 17,000 \text{ kg}, \quad C_{L_{\text{max}}} = 2.2, \quad K = 0.09, \quad C_{D_{\text{0}}} = 0.025 \]
   \[ m_f = 5,000 \text{ kg}, \quad C = 0.84 \text{ lb/hr/lb} \]

   a. How long this aircraft can search for its target?
   b. For this duration, what should be its velocity, if flies at 10,000 ft?

18. The aircraft in problem 4 consumes 10% of its fuel during taxi, take-off and climb and is required to have 20 minutes of reserve fuel. What is its maximum range?

19. The aircraft in problem 7 is required to accomplish a mission at 500 km distance and 20,000 ft altitude. How much fuel this mission needs, if specific fuel consumption is 0.8 lb/hr/lb?

20. The aircraft in problem 4 is flying at 20,000 ft and have a 30 knot headwind. How far this aircraft can fly at this flight condition?

21. Between two aircraft of problems 7 and 17, which one has higher maximum speed at sea level? What about 15,000 ft?

22. The aircraft in problem 4 is required to fly 20% faster at sea level. How much more thrust its engine must produce?

23. Compare maximum lift-to-drag ratio of four aircraft in problems 3, 4, 7, and 17.

24. The aircraft in problem 7 is required to fly with 200 knot at sea level vertically. What is the required engine thrust?
25. Which flight program for aircraft in problem 4 delivers more range? Fly with maximum speed or fly with $1.5V_s$ (both at sea level).

26. What flight program of the aircraft in problem 4 delivers more endurance?

27. Consider the transport aircraft in problem 4 has 200 seats, but carrying only 100 passengers. Calculate maximum range if flying at 30,000 ft. Assume average mass of each passenger is 75 kg and they each carry a 20 kg bag.

28. The stealth aircraft $F-117$ has a takeoff mass of 23814 kg, a wing area of 105.9 m$^2$, and two jet engines, each delivering 48 kN of thrust. If zero-lift drag coefficient is 0.05 and induced drag factor is 0.1, calculate the following:

   a. Maximum speed of this aircraft
   b. Absolute ceiling of this aircraft.

Assume the climb to the absolute ceiling take 7 minutes, and the average engine thrust during climb is equal to 90% of the maximum thrust at sea level. Ignore the effect of the fuel consumed during take-off. $C_{L\text{max}} = 2, C = 0.9 \text{ N/hr/N}$.

29. The fighter aircraft $F/A-18$ (Hornet) has a takeoff mass of 25400 kg, a wing area of 37.16 m$^2$ and two turbofan engines that each producing 71.2 kN of thrust. If $B = 11.43$ m, $AR = 3.5, e = 0.85, C_{Do} = 0.02$

   a. Determine its absolute ceiling. Assume the climb to the absolute ceiling take 15 minutes, and the average engine thrust during climb is equal to 90% of the maximum thrust at sea level. Ignore the effect of the fuel consumed during take-off. $C_{L\text{max}} = 1.8, C = 0.7 \text{ N/hr/N}$.
   b. The mass of aircraft structure is a 10,455 kg. If the pilot mass is 95 kg and the aircraft has only 50 kg of fuel, what is its absolute ceiling?

30. The fighter $Super Etandard$ has a takeoff mass of 12000 kg, a wing area of 28.4 m$^2$ and a jet engine with 49 kN of thrust. Assume it has 5,000 kg of fuel and the following data:

   $AR = 3.23, e = 0.78, C_{Do} = 0.024, C = 0.9 \text{ lb/hr/lb}, b = 9.6 \text{ m}$

Determine:
   a. maximum range
   b. maximum endurance.

31. A designer is designing a jet fighter that is required to have 100,000 ft of absolute ceiling. Its initial data are

   $m_{TO} = 20,000, S = 40 \text{ m}^2, C_{Do} = 0.017, K = 0.06$

Determine how much thrust the engine must be able to produce such that this fighter can fulfill this mission. Assume the climb to the absolute ceiling take 5 minutes, and the average engine thrust during climb is equal to 90% of the maximum thrust at sea level. Ignore the effect of the fuel consumed during take-off. $C_{L\text{max}} = 2.1, C = 1.1 \text{ N/hr/N}$.
32. An experimental aircraft X-31A has a takeoff mass of 6,335 kg, a wing area of 21 m$^2$, with a wing span of 6.95 m, and a turbofan engine with 47.2 kN of thrust. If $C_{D0} = 0.016$ and $e = 0.92$, determine the aircraft absolute ceiling.

33. Determine absolute ceiling of reconnaissance jet aircraft U-2A with the following data:

$m_{TO} = 16,000$ lb, $S = m^2$, $T = 11,200$ lb, $b = 80$ ft, $AR = 11$, $e = 0.94$, $C_{D0} = 0.018$

Assume the climb to the absolute ceiling take 10 minutes, and the average engine thrust during climb is equal to 90% of the maximum thrust at sea level. Ignore the effect of the fuel consumed during take-off. $C_{L_{max}} = 2$, $C = 0.92$ N/hr/N.

34. Determine maximum speed and absolute ceiling of reconnaissance jet aircraft Lockheed SR-71 Blackbird (Figure 5.13) with the following data:

$m_{TO} = 78,000$ kg, $S = 170$ m$^2$, $T = 2 \times 145$ kN, $b = 16.94$ m, $AR = 11$, $e = 0.94$, $C_{D0} = 0.032$

(at supersonic speeds)

Assume the climb to the absolute ceiling take 12 minutes, and the average engine thrust during climb is equal to 90% of the maximum thrust at sea level. Ignore the effect of the fuel consumed during take-off. $C_{L_{max}} = 2.4$, $C = 0.95$ N/hr/N.

35. Determine maximum range of unmanned aircraft Global Hawk with the following data:

$m_{TO} = 11,600$ kg, $S = 50.2$ m$^2$, $T = 31,4$ kN, $b = 35.4$ m, $AR = 22$, $e = 0.94$, $C_{D0} = 0.018$, $C = 0.6$ lbf/hr/lbf, $m_f = 6,590$ kg

36. Derive equation 5.90.

37. A jet aircraft with twin jet engines each generating 8,000 N of thrust has the following characteristics:

$W_{TO} = 16,300$ lbf, $W_f = 4,000$ lbf, $S = 312$ ft$^2$, $K = 0.025$, $C_{D0} = 0.038$, $C_{L_{max}} = 1.2$, $C = 0.65$ lbf/hr/lbf

Calculate the absolute ceiling, if it takes 30 minutes to climb to that altitude with an average fuel consumption of 1800 lbf per hour.

38. Consider a single engine jet aircraft with the following features:

$m_{TO} = 7,400$, kg $S = 110$ m$^2$, $e = 0.92$, $AR = 9.2$, $C_{D0} = 0.024$

Maximum speed at 12,000 ft is 270 KTAS. You are required to add another engine to this aircraft with the same thrust (to have a twin engine). Determine the maximum speed for the twin-engine configuration at 25,000 ft in KEAS.

39. Consider the following jet aircraft that has a fuel capacity of 10,000 kg and specific fuel consumption of 0.7 lb/hr/lb.

$S = 217$ m$^2$, $m_{TO} = 63,000$ kg, $K = 0.043$, $C_{D0} = 0.021$, $C_{L_{max}} = 1.7$

a. What would be the range (in km), if it flies with a constant speed of 160 m/s and constant angle of attack, assuming that the aircraft begins its flight at 18,000 ft altitude.

b. What is the new altitude at the end of this flight?
40. An anti-submarine jet aircraft with the following characteristics is above Atlantic Ocean and is searching for a target submarine.

\[ \text{m}_{\text{TO}} = 17,000 \text{ kg}, \text{m}_\text{i} = 5,000 \text{ kg}, S = 42 \text{ m}^2, \quad C_{\text{Lmax}} = 2.2, \quad C_{\text{Do}} = 0.025, \quad e = 0.87, \quad b = 20 \text{ m}, \quad C = 0.84 \text{ lb/hr/lb} \]

a. Determine the maximum duration (in hour) that this aircraft is able to search for a target.
b. For this duration, what should be the velocity, if flies at 10,000 ft?

41. A jet aircraft has a mission to fly 5,000 km (as its maximum range) at 18000 ft altitude. The aircraft has the following characteristics:

\[ \text{m}_\text{o} = 11,000 \text{ kg}, S = 32 \text{ m}^2, C_{\text{Do}} = 0.021, C = 0.6 \text{ lb/hr/lb}, K = 0.07, T_{\text{maxSL}} = 8000 \text{ N}, C_{\text{Lmax}} = 1.8 \]

a. What percentage of the aircraft initial weight should be fuel weight, in order to perform this mission successfully?
b. How long this mission will take (in hour)?

42. A jet aircraft with the following characteristics has a maximum speed of Mach 0.62 at 12,000 ft altitude (ISA + 15 flight condition). Determine the aircraft new maximum speed (in KTAS) if it employs the afterburner which results in a 30% increase in the maximum engine thrust.

\[ S = 21 \text{ m}^2, \text{m}_{\text{TO}} = 13,000 \text{ kg}, C_{\text{Do}} = 0.029, \text{AR} = 8.5, e = 0.83 \]

Plot the variations of lift-to-drag ratio versus speed for the aircraft in problem 2 when flying at sea level. The speed range is between the stall speed to the maximum speed.

43. Consider the aircraft in problem 40. The aircraft is at 15,000 ft altitude with a speed of 240 KTAS. The fuel mass is 4000 kg. Determine the range,

a. if the pilot holds the altitude and speed constant throughout the flight.
b. if the pilot holds the lift coefficient and speed constant throughout the flight.
c. if the pilot holds the lift coefficient and altitude constant throughout the flight.

44. Consider the aircraft in problem 40. The aircraft is at 25,000 ft altitude with a speed of 180 KTAS. The fuel mass is 4000 kg. Determine is the endurance,

a. if the pilot holds the altitude and speed constant throughout the flight.
b. if the pilot holds the lift coefficient and speed constant throughout the flight.
c. if the pilot holds the lift coefficient and altitude constant throughout the flight.

45. A very large jet transport aircraft with a mass of 600,000 kg, a wing area of 900 m², is equipped with four turbofan engines; and has the following characteristics:

\[ C_{\text{Do}} = 0.025, \quad K = 0.04, \quad T_{\text{SL}} = 4\times300 \text{ kN}, \quad C_{\text{Lmax}} = 2.7 \]

Determine

a. cruise altitude
b. cruising speed in terms of Mach number

Ignore other phases (including the climb phase) of the flight, and assume that this cruise altitude is to maximize the range.
46. A regional jet airliner with a take-off mass of 36,000 kg, a wing area of 83 m², is equipped with two turbofan engines; and has the following characteristics:

\[ C_{D_0} = 0.018, \quad K = 0.035, \quad T_{SL} = 2\times60 \text{ kN}, \quad C_{L_{\text{max}}} = 2.6 \]

Determine
a. cruise altitude
b. cruising speed in terms of Mach number

Ignore other phases (including the climb phase) of the flight, and assume that this cruise altitude is to maximize the range.

47. A business jet aircraft (Cessna Citation) with a take-off mass of 9,000 kg, a wing area of 35 m², is equipped with two turbofan engines; and assume the following characteristics:

\[ C_{D_0} = 0.022, \quad K = 0.042, \quad T_{SL} = 2\times18 \text{ kN}, \quad C_{L_{\text{max}}} = 1.9 \]

Determine
a. cruise altitude
b. cruising speed in terms of Mach number

Ignore other phases (including the climb phase) of the flight, and assume that this cruise altitude is to maximize the range.

48. A small jet aircraft has the following features:
\[ m = 2,400 \text{ kg}, \quad S = 24 \text{ m}^2, \quad C_{D_0} = 0.02, \quad K = 0.05, \quad V_s = 80 \text{ knot} \]

Determine the minimum drag velocity, and the minimum thrust that the jet engine needs to generate at sea level in order for the aircraft to be airborne for a steady level flight at 5000 m.

49. A business jet aircraft has the following features:
\[ W = 90,000 \text{ lb}, \quad S = 400 \text{ ft}^2, \quad C_{D_0} = 0.021, \quad K = 0.045, \quad V_s = 92 \text{ knot} \]

Determine the minimum drag velocity, and the minimum thrust that the jet engine needs to generate at sea level in order for the aircraft to be airborne for a steady level flight at 30,000 ft.
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